A robust $L_1$ controller design for continuous-time TS systems with persistent bounded disturbance and actuator saturation

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ABSTRACT

This paper proposes a novel approach for $L_1$ performance controller design of saturated input nonlinear systems described by Takagi-Sugeno (TS) fuzzy systems. The proposed method utilizes parallel distributed compensation (PDC) approach to minimize the effect of persistent bounded disturbances on the system output and the design conditions are formulated in terms of linear matrix inequalities (LMIs) and a generalized eigenvalue problem (GEVP). The novelty of this paper is to design a saturated robust $L_1$ PDC controller whose procedure comprises new techniques for both minimizing the $L_1$ performance gain and handling actuator saturation constraint. These techniques lead to less conservative results compared to recently published papers in the viewpoint of $L_1$ performance criterion gain as well as the control input saturation. The advantages of the proposed approach in each above-mentioned viewpoint are shown via numerical examples by comparing our results with the existing literatures. Finally, a nonlinear electromagnetic suspension (EMS) system is considered to verify the applicability and efficiency of this novel design method.

1. Introduction

Recently, controller design of nonlinear systems is at the center of attention of many researchers, due to the nonlinear and complex nature of industrial applications. Among the available control strategies, the simplicity and effectiveness of fuzzy control make this method a very active research field (Al-Hadithi et al., 2015). Firstly, model-free fuzzy control scheme was proposed which was obtained based on heuristic and expert human knowledge (Sadeghi et al., 2015). However, lack of systematic procedure to guarantee the stability and desirable performance issues, encourage researchers to develop Takagi-Sugeno (TS) fuzzy model-based control (Sha Sadeghi et al., 2014; Nikdel et al., 2014).

Recently, TS model-based control has been widely used for stability analysis and controller synthesis of complex nonlinear ordinary and partial differential systems (Tanaka, 2001; Sha Sadeghi et al., 2014; Marouf et al., 2016). Parallel distributed compensation (PDC) and non-PDC control structures provide systematic approaches to accomplish the stabilization and performance issues for such TS models in terms of linear matrix inequalities (LMIs) (Scherer and Weiland, 2004; Boyd et al., 1994). The existence of external disturbance and actuator saturation in industrial applications makes the disturbance alleviation problem and stability of the closed-loop systems with saturated input attractive research topics.

There has been a vast of research activities on the TS fuzzy based disturbance rejection which consider energy bounded $L_2$ disturbances (Asemani and Majd, 2015; Saifia et al., 2012; Sadeghi et al., 2015; Khooban et al., 2016). However, these approaches are not applicable for the external disturbances that are persistent bounded and belong to $L_\infty$ space (Jhi and Tseng, 2012). In this case, $L_1$ performance criterion should be considered to design a controller, which minimizes the amplitude of the persistent bounded disturbance with respect to the system output (Ahn et al., 2016).

The $L_1$ performance criterion in nonlinear systems has been investigated in the literature for TS fuzzy observers and filters (Mohammadian et al., 2015) and robust controllers (Mahmoud and Shi, 2011; Salcedo and Martínez, 2008; Han and Chang, 2014; Jafarnejadsani and Pieper, 2015). In Mahmoud and Shi (2011), the persistent bounded disturbance minimization problem is restated as a generalized eigenvalue problem (GEVP) by the tools of invariant set analysis. In Salcedo and Martínez (2008), the $L_1$ performance controller is designed by utilizing the S-procedure and elimination lemma (Scherer and Weiland, 2004). The approach of Salcedo and Martínez (2008) is extended to dynamic output feedback control (Han and Chang, 2014) and gain-scheduled control (Jafarnejadsani and Pieper, 2015). However, in the aforementioned works, physical restrictions on employing amplitude-bounded inputs have not been addressed.

Actuator saturation is one of the most usual nonlinearity sources in
control engineering due to physical restriction of applying large control signals (Kao et al., 2015). Actuator saturation can severely decrease the closed-loop system performance and sometimes may lead to the closed-loop instability (Wu and Lin, 2014). Motivated by this fact, a great deal of effort has been focused on the TS systems with saturated control inputs (Nguyen et al., 2015).

Few TS-based works investigated the actuator saturations. In Bezzaoucha et al. (2013), the authors utilize the TS model of the saturation function to combine the bounded constraints with the control synthesis. However, the closed-loop system was derived in an affine form. The presented approach in Bezzaoucha et al. (2013), can only guarantee the boundedness and convergence of the states to an origin-centered ball without achieving asymptotic stability. In Cao and Lin (2003), the closed-loop system with saturated input is restated by the so-called polytopic representation. Then, an invariant set framework is developed such that every state trajectory that starts inside this invariant set will not surpass it and the states will converge to their equilibrium points (Cao and Lin, 2003). Different approaches are proposed based on this idea (Zhao and Gao, 2012; Benzaouia et al., 2015; Yang and Tong, 2015; Chang and Shih, 2015). In Du and Zhang (2009), TS fuzzy state feedback controller design for electrohydraulic active vehicle suspensions has been presented by considering actuator saturation. A method for designing robust $H_{\infty}$ static output feedback stabilization of TS fuzzy systems under actuator saturation is presented in Saïfa et al. (2012). In Zhao and Gao (2012), a PDC controller is designed for overhead crane with input delay and actuator saturation. Fault tolerant saturated control problem for discrete-time TS fuzzy systems with delay and input constraint saturation is studied in Benzaouia et al. (2015). In Yang and Tong (2015), output feedback robust stabilization problem of a class of switched fuzzy systems with immeasurable states and actuator saturation has been considered. However, these approaches are only applicable for non-perturbed TS systems (Zhao and Gao, 2012; Yang and Tong, 2015; Zhao and Li, 2015) or TS systems with $L_1$ spaced disturbance inputs (Saïfa et al., 2012; Chang and Shih, 2015).

Recently, in Nguyen et al. (2015), the controller design of input saturated TS systems with amplitude-bounded disturbance is studied. The actuator saturation is handled by a so-called anti windup (AW) scheme. In the AW method that can be considered as a cascade structure, two controllers are designed. Based on an inner loop feedback linearization controller, an outer loop is designed based on the linear control theory (Güfner et al., 2012). However, the controller design conditions of Nguyen et al. (2015) are derived in terms of LMIs by pre-choosing of two scalars. Unfortunately, no systematic approach has been presented for selecting the parameters. Therefore, one must choose these parameters by trial and error. Unproper pre-selections of these parameters may lead to non-optimum solutions and even infeasible results.

Nowadays, electromagnetic suspension (EMS) technology is widely used in various applications including frictionless bearings, ultra-precision motion platforms, nano-scale positioning systems, vibration isolation, levitation of molten in induction furnaces, levitation of wind precision motion platforms, nano-scale positioning systems, vibration...
\[ \pi(t) = \text{sat}(u(t)) = \begin{cases} -u_{\text{lim}} & \text{if } u(t) < -u_{\text{lim}} \\ u(t) & \text{if } -u_{\text{lim}} \leq u(t) \leq u_{\text{lim}} \\ u_{\text{lim}} & \text{if } u(t) > u_{\text{lim}} \end{cases} \]

with \( u_{\text{lim}} \) be the control input limit. Furthermore, \( h_j(z(t)) \) are the normalized membership functions which satisfy the convex sum property:

\[ \sum_{j=1}^{r} h_j(z(t)) = 1 \]  

By adding the null term \( \sum_{j=1}^{r} h_j(z(t))B_1 \frac{1+e}{2}(u(t)-v(t)) \) with \( 0 < e < 1 \) to (1), one has:

\[ x(t) = \sum_{j=1}^{r} h_j(z(t)) \left\{ A_j x(t) + B_1 \frac{1+e}{2} u(t) + B_2 \left( \pi(t) - \frac{1+e}{2} u(t) \right) + E_j x(t) \right\} \]

\[ = \sum_{j=1}^{r} h_j(z(t)) \left\{ A_j x(t) + B_1 \frac{1+e}{2} u(t) + B_2 \left( \pi(t) + E_j x(t) \right) \right\} \]

where \( r(t) = \pi(t) - \frac{1+e}{2} u(t) \). The external disturbance \( v(t) \) is persistent and generally does not converge to zero when \( t \to \infty \) (Salcedo and Martínez, 2008).

### 2.2. Lyapunov function and state feedback controller

For persistent bounded disturbances, one can guarantee the boundedness of the system state vector as \( t \to \infty \). Therefore, the stabilization conditions are derived through bounded input-bounded output (BIBO) stability schemes. To obtain BIBO conditions, quadratic Lyapunov function (QLF) and PDC controller are considered as follows:

\[ V_\beta(x(t)) = x(t)^T P^{-1} x(t) \]

\[ u(t) = \sum_{j=1}^{r} h_j(z(t)) F_j P^{-1} x(t) \]

where \( P = P^T > 0 \) and \( F_1, F_2 \) are the local feedback gains. By substituting the PDC controller (6) into the open-loop TS system (4), the closed-loop TS system will be obtained as:

\[ \dot{x}(t) = \sum_{j=1}^{r} \sum_{j=1}^{r} h_j(z(t)) h_k(z(t)) \left\{ A_j P + B_1 \frac{1+e}{2} B_j F_k \right\} P^{-1} x(t) + \sum_{j=1}^{r} h_j(z(t)) [B_2 F_j P^{-1} x(t) + E_j x(t)] \]

The following Lemmas are required for the main results of this paper.

**Lemma 1.** (Du and Zhang, 2009): For the saturation constraint defined by (2), as long as \( |u(t)| < \frac{u_{\text{lim}}}{e} \), one has:

\[ r(t) v(t) = \left( \pi(t) - \frac{1+e}{2} u(t) \right)^T \left( \pi(t) - \frac{1+e}{2} u(t) \right) \leq \left( \frac{1-e}{2} \right) u^T(t) u(t) \]

**Lemma 2.** (Tuan et al., 2001): Inequality \( \sum_{j=1}^{r} \sum_{j=1}^{r} h_j(z(t)) h_k(z(t)) y_{ij} < 0 \) is satisfied if:

\[ \left\{ \begin{array}{l} \gamma_i < 0 \\ \gamma_i + \gamma_j < 0 \quad \text{for } i \neq j < 1, ..., r \end{array} \right. \]

**Lemma 3.** (Tseng et al., 2009): If a real scalar function \( S(t) \) satisfies the differential inequality \( \dot{S}(t) \leq -\alpha S(t) + \beta \psi(t) \) with \( \alpha \) and \( \beta \) being positive scalar, then:

\[ S(t) \leq e^{-\alpha t} S(0) + \beta \int_{0}^{t} e^{-\alpha \tau} \psi(t - \tau) d\tau \]

In this paper, the problem is to derive BIBO analysis and robust \( L_1 \) controller synthesis conditions in terms of LMIs such that the effect of peak of persistent bounded disturbance \( v(t) \) on the peak of system outputs will be minimized and the input saturation constraint will be satisfied.

**Notation:** For brevity, in the following, \( h_i, x, \dot{x}, u, v, y, r \) and \( M_i \) denote \( h_j(z(t)), x(t), \dot{x}(t), u(t), v(t), y(t), r(t) \) and \( \sum_{j=1}^{r} h_j(z(t)) M_i \) respectively.

### 3. Main results

In this section, first we will present the \( L_1 \) performance criterion. Then, a new theorem will be proposed to minimize the \( L_1 \) performance upper bound of the closed-loop TS fuzzy model (7) in the presence of input saturation.

The optimal \( L_1 \) performance problem is to design the PDC controller (6), such that the following problem is guaranteed for the closed-loop TS system (7):

\[ \min_{\pi(t)} \| \pi(t) \|_{\infty} = \min_{\pi(t)} \| \pi(t) \|_{\infty} \]

where the infinity norm of a vector signal is defined as Salcedo and Martínez (2008):

\[ \| \pi(t) \|_{\infty} = \sup_{v \geq 0} \frac{\| \pi(t) \|}{\| v \|} \]

In the definition (11), ratio of the infinity norm of the output to the infinity norm of the disturbance is considered.

In the following, sufficient conditions will be derived in terms of LMIs such that the closed-loop TS system (7) guarantees the \( L_1 \) performance criterion (11).

**Theorem 1.** For given scalars \( \alpha > 0, r_1 \geq 0 \) and \( r_2 \geq 0 \), if there exist symmetric matrix \( P \) and matrices \( F_i \) for \( i = 1, ..., r \) such that the following LMIs are satisfied:

\[ \left\{ \begin{array}{l} Q_{ij} < 0 \quad \text{for } i \leq r \\ \frac{1}{2} Q_{ij} + Q_{ji} + Q_{ii} < 0 \quad \text{for } i < j \leq r \\ \alpha P \quad 0 \quad PC_i \quad 0 \\ 0 \quad (Q - \beta P) D_j \quad D_j^T \quad 0 \\ CP \quad D_i \quad 0 \quad \Omega \end{array} \right. \]

**Proof.** Consider the time derivative of QLF (5) as:

\[ V_\beta \dot{V}_\beta = x(t)^T P^{-1} \dot{x}(t) \]

Substituting the closed-loop system dynamics (7) in (16) yields:
\[ V_0 = x^T P^{-1} \begin{bmatrix} A \rho + \frac{1 + \epsilon}{2} B F \end{bmatrix}^T P^{-1} x + x^T P^{-1} \begin{bmatrix} A \rho + \frac{1 + \epsilon}{2} B F \end{bmatrix} P^{-1} x + \beta v^T v \]

Consequently, the LMI conditions

\[ P^{-1} + \beta v^T v \]

where \( I \) is the identity matrix with appropriate dimensions. In the following, sufficient conditions to guarantee the negative definiteness of \( \Gamma \) are derived in terms of LMI.

\[ \Gamma = P^{-1} + \beta v^T v \]

where

\[ \Gamma_i = \begin{bmatrix} \text{sym}(A \rho + \frac{1 + \epsilon}{2} B F) + \alpha P B^T \rho \end{bmatrix} \begin{bmatrix} 0 \quad 0 \quad 0 \quad -\beta I \end{bmatrix} \]

As can be seen, the matrix \( \Gamma_i \) in (19) has a zero diagonal element. Consequently, the negative definiteness of \( \Gamma_i \) cannot be guaranteed. In order to solve this problem, we use Lemma 1 and S-procedure (Boyd et al., 1994). Substituting the PDC controller (6) into (8) results in:

\[ r^T v(t) \leq \left( \frac{1 + \epsilon}{2} \right)^2 x^T P^{-1} F_i F_r P^{-1} x \]

Inequality (20) can be restated as:

\[ \begin{bmatrix} P^{-1} r \end{bmatrix} \begin{bmatrix} \left( \frac{1 + \epsilon}{2} \right)^2 F_i F_r \end{bmatrix} \begin{bmatrix} P^{-1} r \end{bmatrix} \geq 0 \]

By employing S-procedure (19) subject to (21), one has:

\[ \begin{bmatrix} P^{-1} r \end{bmatrix} \begin{bmatrix} \text{sym}(A \rho + \frac{1 + \epsilon}{2} B F) + \alpha P B^T \rho \end{bmatrix} \begin{bmatrix} 0 \quad 0 \quad 0 \quad -\beta I \end{bmatrix} \begin{bmatrix} P^{-1} r \end{bmatrix} < 0 \]

where \( \tau \geq 0 \). Employing congruence lemma (Scherer and Weiland, 2004) and Schur complement (Scherer and Weiland, 2004) on (22), provides:

\[ \begin{bmatrix} \text{sym}(A \rho + \frac{1 + \epsilon}{2} B F) + \alpha P B^T \rho \end{bmatrix} \begin{bmatrix} 0 \quad 0 \quad 0 \quad -\beta I \end{bmatrix} \begin{bmatrix} \text{sym}(A \rho + \frac{1 + \epsilon}{2} B F) + \alpha P B^T \rho \end{bmatrix} < 0 \]

Based on Lemma 2, the negative definiteness of (23) is enforced if (13) is satisfied. Therefore:

\[ V_0 \leq -\alpha x^T P^{-1} x + \beta v^T v = -\alpha V_0 + \beta v^T v \]

By considering Lemma 3, one obtains:

\[ x^T P^{-1} x \leq e^{-\alpha x^T F_i F_r P^{-1} x} \int_0^t e^{-\alpha v^T v(t)} dt \]

In order to derive an inequality in the form of (11), consider the following inequality:

\[ y^T y \leq \frac{c}{x^T} \int \begin{bmatrix} 0 \quad 0 \quad 0 \quad 0 \end{bmatrix} y \]

In the following, we derive the sufficient conditions for satisfying (27). Recall the system output (1). Inequality (27) is implied by:

\[ \begin{bmatrix} x \end{bmatrix}^T \begin{bmatrix} C_1 & D_1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \leq \begin{bmatrix} x \end{bmatrix}^T \begin{bmatrix} a P^{-1} & 0 \end{bmatrix} \begin{bmatrix} \Omega - \beta I \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} C_1 & D_1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} > 0 \]

Equivalently, one has:

\[ \begin{bmatrix} x \end{bmatrix}^T \left( a P^{-1} - \frac{1}{\Omega} C_1 D_1^T C_1 \right) \begin{bmatrix} x \end{bmatrix} > 0 \]

Considering congruence lemma and Schur complement provides:

\[ \begin{bmatrix} a P^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & C_1 \end{bmatrix} \begin{bmatrix} 0 & \Omega - \beta I \end{bmatrix} D_1^T \begin{bmatrix} x \end{bmatrix} > 0 \]

By pre- and post-multiplying (30) by \( \text{diag}[P, I, I] \) and its transpose, respectively, LMI (14) will be obtained. Based on (27), one has:

\[ y^T y \leq \Gamma_i \leq \Omega (\alpha x^T P^{-1} x + (\Omega - \beta v^T v) \]

Substituting (26) in (31) and considering the definition (12) result in:

\[ y^T y \leq \Omega^2 \|v(t)\|^2 \]

Therefore, the L1 performance (11) will be guaranteed. On the other hand, the inequality (8) of Lemma 1 is satisfied for the domain \( \mathcal{R}(F, P) = \{x \mid |x| \leq P^{-1} x < \frac{2}{\alpha} \} \). Consequently, the LMI conditions for satisfying this domain must be derived. Based on the set inverse analysis presented in Cao and Lin (2003) and considering the same procedure discussed in Du and Zhang (2009), the condition which guarantees the ellipsoid \( \mathcal{R}(\rho, P) = \{x \mid x^T P^{-1} x < \rho \} \) being inside the domain \( \mathcal{R}(F, P) \), i.e. \( \mathcal{R}(\rho, P) \subset \mathcal{R}(F, P) \), is given by:

\[ F_{ik}^T P^{-1} P^{-1} F_{ik} \leq \frac{\rho}{\alpha} \]

where \( F_{ik} \) indicates the \( k \)-th row of \( F \). By employing Schur complement on the condition (33), LMI (15) is obtained. The proof is completed.

**Remark 1.** In Theorem 1, we utilized the definition (11) which makes the design procedure more complicated. If one considers only the procedure given in the proof of Theorem 1 up to inequality (25), then the L1 performance criterion respect to the state vector will be achieved. One of the drawbacks of the formulations presented in Tseng et al. (2009) is that the procedure of performance gain minimization is not straightforward since different parameters should be optimized simultaneously. This optimization cannot be easily restated in term of GEVP.
Remark 2. In deriving (26), it is assumed that the initial condition is zero. However, the LMIs (13) guarantee the closed-loop stability of the non-perturbed TS system. To prove this, consider (22). The negative definiteness of (22) guarantees the following inequality:

\[
[p^{-1} - r] ^T \text{sym}(A_P + \frac{1+\varepsilon}{2} B F_i) + a P + \varepsilon(\frac{1-e^{-\varepsilon}}{2}) F_i^T F_i - \varepsilon f_i r < 0
\]

(34)

If (34) holds, the Lyapunov inequality \(V + aV < 0\) for non-perturbed closed-loop TS system (7) is satisfied. Therefore, the exponential stabilization of TS model with saturated input is achieved.

Remark 3. In this paper, we address simultaneously the problem of stabilization of the persistent bounded disturbed TS systems with saturated control input. However, it should be noted that the design procedures for satisfying L1 performance criterion and actuator saturation constraint are both novel procedures and provide less conservative LMI conditions. To verify this, two corollaries will be presented in the following based on the proof procedure of Theorem 1. In Corollary 1, new sufficient conditions for perturbed closed-loop TS systems with persistent bounded disturbances will be proposed. In other words, the conditions of Theorem 1 are detracted to L1 performance criterion conditions without input constraint in this corollary. Meanwhile, in Corollary 2, new conditions for the stability of the non-perturbed closed-loop system with input saturation are presented.

Corollary 1. The closed-loop TS system (4) without any limits on the control signal guarantees the L1 performance (11) with the attenuation level \(\Omega\), if for a given scalar \(a > 0\), there exist symmetric positive definite matrix \(P\) and matrices \(F_i\) for \(i = 1, ..., r\), such that the following conditions hold:

\[
\begin{align*}
W_{ii} &< 0 \quad \text{for } i \leq r \\
\frac{1}{r} \sum_{i=1}^{r} W_{ii} + W_{ij} &< 0 \quad \text{for } i < j \leq r
\end{align*}
\]

(35)

\[
\begin{bmatrix}
aP & 0 \\
0 & (\Omega - \beta I) D_i^T & D_i \end{bmatrix} > 0
\]

(36)

where

\[
W_i = \begin{bmatrix}
sym(A_P + B F_i) + a P & E_i \\
E_i^T & -\beta I
\end{bmatrix}
\]

Proof. Here, the input saturation constraint is not considered. In this case, the closed-loop TS system is restated as:

\[
\dot{x} = (A_P + B F_i) x + E_i y
\]

(37)

By substituting the new closed-loop system (37) into the time derivative of the Lyapunov function (16) and considering the same procedure as presented in the proof of Theorem 1 in which (17) leads to (24), one concludes that:

\[
\Gamma = [p^{-1} - r]^T \begin{bmatrix}
sym(A_P + B F_i) + a P & E_i \\
E_i^T & -\beta I
\end{bmatrix} [p^{-1} - r] < 0
\]

(38)

Based on Lemma 2, inequality (38) leads to (35). Moreover, by considering the same procedure as applied in (25) to derive (32), (36) will be achieved. The proof is completed.

Corollary 2. Consider the closed-loop TS system (7) with \(v(t) = 0\). If for a given scalar \(\rho > 0\), there exist symmetric positive definite matrix \(P\) and matrices \(F_i\) for \(i = 1, ..., r\), such that the following conditions hold:

\[
\begin{align*}
W_{ii} &< 0 \quad \text{for } i \leq r \\
\frac{1}{r} \sum_{i=1}^{r} W_{ii} + W_{ij} &< 0 \quad \text{for } i < j \leq r
\end{align*}
\]

(39)

\[
\begin{bmatrix}
\frac{\Omega \rho^2}{4} F_i^T F_i & P^T P \\
P^T F_i & 0
\end{bmatrix} > 0
\]

(40)

where \(F_i^T\) indicates the \(k\)-th row of \(F_i\) and \(W_i\) is defined as

\[
\begin{bmatrix}
\text{sym}(A_P + \frac{1+\varepsilon}{2} B F_i) & B_i (\frac{1-e^{-\varepsilon}}{2}) F_i^T \\
B_i^T & -\varepsilon f_i I & 0 \\
(\frac{1+\varepsilon}{2}) F_i & 0 & -\varepsilon f_i I
\end{bmatrix}
\]

Then, the non-perturbed closed-loop TS system (7) is stable for the local region \(\rho(\rho, P) = |\varepsilon|^T P^{-1} \chi < \rho\).

Proof.: Since here we do not consider the effect of persistent bounded disturbance, the non-perturbed TS system will be obtained by letting \(E_i = 0\) for \(i, ..., r\) in (7):

\[
\dot{x} = \left(A_P + \frac{1+\varepsilon}{2} B F_i\right) P^{-1} x + B_i y
\]

(41)

By substituting the closed-loop TS system (41) into the time derivative of the Lyapunov function (16) and then employing the S-procedure on the Lyapunov inequality subject to the condition of Lemma 1, (39) will be obtained. As discussed in the proof of Theorem 1, (15) is a sufficient condition for \(\mu(t) < \frac{\rho}{2}\) given in Lemma 1 and it should be considered in Corollary 2. The proof is completed.

Remark 4. In Theorem 1 and Corollaries 1 and 2, three different controller design methods have been proposed. In the following, we provide a procedure to construct the fuzzy controller for nonlinear systems:

1. Represent the dynamics of the nonlinear system in the form of TS fuzzy model (1) which consists of the local system matrices \(A_i, B_i, C_i\) and \(E_i\) and the normalized membership functions \(h_i(\varepsilon(t))\).
2. Consider one the following methods based on the practical situation of the nonlinear system:
   a. Solve LMIs (13)–(15) of Theorem 1 to find the controller gains \(F_i\) and the Lyapunov matrix \(P\) for persistent bounded disturbed and input saturated nonlinear systems.
   b. Solve LMIs (35)–(36) of Corollary 1 to find the controller gains \(F_i\) and the Lyapunov matrix \(P\) for persistent bounded disturbed and non-saturated nonlinear systems.
   c. Solve LMIs (39)–(40) of Corollary 2 to find the controller gains \(F_i\) and the Lyapunov matrix \(P\) for non-disturbed and input saturated nonlinear systems.
3. Use the computed LMI decision variables \(F_i\) and \(P\) obtained in the step 2 and the normalized membership functions \(h_i(\varepsilon(t))\) derived in the step 1 to establish the fuzzy PDC controller (6).

4. Simulation

In this section, four examples are presented to verify the advantages and merits of the proposed approach. In Example 1, the goal is to show the advantage of deriving controller design conditions with less number of pre-chosen scalars. To do this, Corollary 1 and Salcedo and Martinez (2008) are considered to calculate the lower bound for L1 performance gain. In addition, the calculation time of each method is given. In Example 2, the conservativeness of the controller design conditions is investigated. Corollary 2 is compared with Du and Zhang (2009) and Nguyen et al. (2015). The region of attraction
Consider a three-rule TS fuzzy system with the following specifications:

\[
\begin{align*}
A_1 &= \begin{bmatrix} 0.59 & -10.29 & 0.1 \\ 0.01 & -2.68 & 0 \\ 0 & -0.17 & -3.11 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -6.02 & 6.64 & 0 \\ 0.35 & -4.21 & 0 \\ 0 & -0.08 & -4.05 \end{bmatrix}, \\
A_3 &= \begin{bmatrix} -14 & -6.33 & 0.5 \\ 0 & 0.05 & 0 \\ -2.05 & 0 & 1.21 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} -1 & 0.02 \\ 1 & 2.18 \\ -1 & 2.87 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} -1 & 0.5 \\ 1.01 & 1 \\ -0.89 & 0 \end{bmatrix}, \\
E_1 &= \begin{bmatrix} 2.45 \\ 1.3 \\ 1 \end{bmatrix}, \\
E_2 &= \begin{bmatrix} 0 \\ 0 \\ 1.62 \end{bmatrix}. 
\end{align*}
\]

In this example, we employ Corollary 1 and Salcedo and Martínez (2008) in order to investigate the advantages of the proposed procedure for computing $L_1$ performance gain. Since in Corollary 1 and Salcedo and Martínez (2008) some scalars may be chosen in prior, therefore, a line search is needed to choose the best values for these parameters. In practice, the optimal values of the pre-fixed parameters, which lead to the global minimum of $L_1$ performance gain, are unknown. Thus, the design methods should be employed for a wide range of scalars variations. However, in this example, we consider moderate and small ranges for scalars variations. In Corollary 1 and Salcedo and Martínez (2008), a line search for $0 < a < 5$ and $0 < \sigma < 20$ with resolution 0.1 is employed. Table 1 demonstrates the best value of scalars $a$ and $\sigma$, and the calculated $L_1$ gain $\Omega$ for Corollary 1 and Salcedo and Martínez (2008). Using the LMI s of Corollary 1, the $L_1$ performance gain is calculated faster than Salcedo and Martínez (2008), especially in the case of the line search in wide areas. In Corollary 1, one dimension search is needed. Meanwhile, in Salcedo and Martínez (2008) two dimensions search is mandatory. Furthermore, Table 1 indicates that Corollary 1 calculates a lower $L_1$ performance bound compared to Salcedo and Martínez (2008). From this, one infers that the proposed procedure for minimization of the $L_1$ performance gain obtains less conservative results compared to the idea of Salcedo and Martínez (2008).

Example 2. (Comparison of conservativeness of conditions): Consider a two-rule TS fuzzy model of the form (1) with the following system matrices and membership functions:

\[
h_1 = 0.59, h_2 = 0.01, a = 0.01, \quad B_1 = \begin{bmatrix} -1 & 0.02 \\ 1 & 2.18 \\ -1 & 2.87 \end{bmatrix}, \\
B_2 = \begin{bmatrix} -1 & 0.5 \\ 1.01 & 1 \\ -0.89 & 0 \end{bmatrix}, \\
E_1 = \begin{bmatrix} 2.45 \\ 1.3 \\ 1 \end{bmatrix}, \\
E_2 = \begin{bmatrix} 0 \\ 0 \\ 1.62 \end{bmatrix}. 
\]

In this example, the goal is to investigate the conservativeness of the novel procedure for considering input saturation constraint in this paper and compare those of the works within the same topic (i.e., Du and Zhang (2009) and Nguyen et al. (2015)). It should be noted that, in Du and Zhang (2009) the controller design conditions are presented for non-perturbed TS systems with input saturation. Therefore, a non-perturbed system (42) is considered and Corollary 2, Du and Zhang (2009) and Nguyen et al. (2015) are applied to the mentioned system.

Let $\varepsilon = 0.9$, $u_{\text{min}} = 10$, $\rho = 5$, and $r = 100$ in Corollary 2 and the approach of Du and Zhang (2009). In addition, set $\delta = 0.01$, $t_i = 0.5$, and $u_{\text{min}} = 10$ in Theorem 1 of Nguyen et al. (2015). Fig. 1 demonstrates the feasibility region of these approaches. The feasible points derived based on Du and Zhang (2009), Nguyen et al. (2015), and Corollary 2 are shown by the marks “\(\text{O}\)” and “+”, and “\(\text{O}\) and +”, respectively. Fig. 1 reveals that the conditions of Corollary 2 provide larger feasible area compared to those of Du and Zhang (2009) and Nguyen et al. (2015). Consequently, one concludes that the proposed approach in this paper leads to the less conservative results.

Example 3. (Estimation of the region of attraction and $L_1$ performance): Consider the following nonlinear open-loop unstable system (Nguyen et al., 2015):

\[
\begin{align*}
\dot{x}_1 &= -x_1 + (0.1 + 0.12x_1^2)x_2 + (1.48 + 0.16x_2^2)w + 0.1w \\
\dot{x}_2 &= x_1 + 0.1w 
\end{align*}
\]

By considering a local region $x_1^2 + x_2^2 < 1.5$, the following equivalent four rule TS fuzzy model will be achieved (Nguyen et al., 2015):

\[
\begin{align*}
A_1 &= \begin{bmatrix} -1 & 0.1 \\ 1 & 0 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0.94 \\ 0 \end{bmatrix}, \\
E_1 &= \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -1 & 0.37 \\ 1 & 0 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 0.94 \\ 0 \end{bmatrix}, \\
E_2 &= \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}. 
\end{align*}
\]

In this example, the goal is to compare our proposed method to the recently published paper in which the TS systems with persistent bounded disturbance and input saturation are considered. Let $u_{\text{min}} = 10$, $\rho = 40$, $\varepsilon = 0.95$, and $r = 1$ in Theorem 1. The controller of Nguyen et al. (2015) is designed based on a tradeoff between the size domain of attraction and the $L_1$ performance gain. Therefore, we consider two cases of achieving a largest region of attraction and a highest $L_1$ performance $\Omega$ of Nguyen et al. (2015), respectively. The $L_1$ performance gain provided by Theorem 1 and Nguyen et al. (2015) are presented in Table 2.

As it can be seen in Table 2, our approach provides a better $L_1$ performance criterion compared to Nguyen et al. (2015). Furthermore, the region of attractions derived by Theorem 1 and Nguyen et al. (2015) are plotted in Fig. 2. In this figure, $R_1$ and $R_2$ stand for the local region of attractions of Theorem 1 and Nguyen et al. (2015), respectively. In addition, $R_1$ shows the region in which the TS model is
valid (i.e. $|x_2| < 1.5$). The regions $R_1$ and $R_2$ must be within the $R_3$. Fig. 2 reveals that, although the $R_1$ does not completely cover the $R_3$, Theorem 1 provides a larger region of attraction compared to Nguyen et al. (2015).

**Example 4. ( Electromagnetic Suspension System):** In this example, the nonlinear dynamic of the EMS MAGLEV train and the equivalent TS fuzzy system are presented. Then, the PDC controller (6) is designed based on its TS fuzzy system. Finally, the designed saturated controller (2) is applied to the original nonlinear EMS.

### A. Nonlinear Dynamic of EMS MAGLEV Train

The model of EMS maglev train comprises a rail, a U-shaped iron core and DC control coils. The EMS schematic model is shown as Fig. 3. The DC control coils are used to keep the position of suspension air gap stable. If the position of suspension air gap changes, controller injects current into the DC control coils and manipulates the suspension force to maintain the position. This leads to the variation of magnetic flux density in the U-shaped iron core. By using Newton’s law and Kirchhoff’s law, dynamic of EMS system can be described as Su et al. (2014):

\[
\begin{align*}
\frac{d^2x(t)}{dt^2} &= -F(i(t), \delta(t)) + f_d(t) + mg \\
F(i(t), \delta(t)) &= \mu_0 N_s N_d A \left( \frac{d\Phi(i(t))}{dt} \right)^2 \\
\Phi(i(t), \delta(t)) &= \delta(t) - R_m i(t) \\
\Psi(i(t), \delta(t)) &= \frac{\mu_0 N_s A}{2}\left( \frac{d\Phi(i(t))}{dt} \right) 
\end{align*}
\]  
(45)

where the parameters of EMS dynamic are presented in Table 3.

### A. Equivalent TS Fuzzy System of EMS MAGLEV Train

The electromagnet system must be kept at the stable position $\delta_{\text{ref}}$. Therefore, an offset input $v_{\text{ref}} = \frac{2mg}{\mu_0 N_s A} \delta_{\text{ref}}$ is needed. Dynamic Eq. (45) can be transformed in such way that the equilibrium point will be in the origin and can be rearranged in the form of nonlinear state space equation (Su et al., 2014):

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \frac{2\sqrt{\delta_{\text{ref}} + \delta_{\text{ref}}(i(t))}}{\delta_{\text{ref}}(i(t)) + \delta_{\text{ref}}} - \frac{(\delta_{\text{ref}} + \delta_{\text{ref}}(i(t))}{\delta_{\text{ref}}(i(t)) + \delta_{\text{ref}}} + \frac{1}{\mu_0 N_s A}f_d(t) \\
\dot{\delta}(t) &= \delta(t) - \delta_{\text{ref}} \\
\dot{\delta}_{\text{ref}}(i(t)) &= \delta(t) - \delta_{\text{ref}} \\
\dot{i}(t) &= \mu_0 N_s A \delta_{\text{ref}}(i(t)) - \frac{2mg}{\mu_0 N_s A}x_1(t) - u(t) 
\end{align*}
\]  
(46)

where $\delta_{\text{ref}}$ is the reference input and $u(t) = v(t) - \delta_{\text{ref}}$.

\[
\begin{align*}
x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
\dot{x}(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
\delta(t) &= \delta(t) - \delta_{\text{ref}} \\
\delta_{\text{ref}}(i(t)) &= \delta(t) - \delta_{\text{ref}} \\
i(t) &= i(t) - \delta_{\text{ref}} \\
u(t) &= u(t) - \delta_{\text{ref}} 
\end{align*}
\]  
(47)

The four-rule TS fuzzy system is obtained with the following matrices and membership functions (Su et al., 2014):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition of parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>Gross mass of carriage and electromagnet</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>Suspension air gap</td>
</tr>
<tr>
<td>$F(i(t), \delta(t))$</td>
<td>Suspension force</td>
</tr>
<tr>
<td>$f_d(t)$</td>
<td>Vertical disturbance force</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational acceleration</td>
</tr>
<tr>
<td>$\mu_a$</td>
<td>Permeability of air</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of turns in the coil</td>
</tr>
<tr>
<td>$a_r$</td>
<td>Valid pole area of the coil</td>
</tr>
<tr>
<td>$i(t)$</td>
<td>Current of the electromagnet coil</td>
</tr>
<tr>
<td>$\Psi(i(t), \delta(t))$</td>
<td>Magnetic potential</td>
</tr>
<tr>
<td>$\nu(t)$</td>
<td>Voltage of the electromagnet coil</td>
</tr>
<tr>
<td>$R_m$</td>
<td>Coil resistance</td>
</tr>
</tbody>
</table>

### Table 3

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition of parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>Gross mass of carriage and electromagnet</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>Suspension air gap</td>
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<tr>
<td>$F(i(t), \delta(t))$</td>
<td>Suspension force</td>
</tr>
<tr>
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<td>Vertical disturbance force</td>
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<td>$g$</td>
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</tr>
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<td>Voltage of the electromagnet coil</td>
</tr>
<tr>
<td>$R_m$</td>
<td>Coil resistance</td>
</tr>
</tbody>
</table>
Table 4
Parameters values for simulation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>150 kg</td>
<td></td>
</tr>
<tr>
<td>( R_m )</td>
<td>1.1 ( \Omega )</td>
<td></td>
</tr>
<tr>
<td>( g )</td>
<td>9.8 ( m/s^2 )</td>
<td></td>
</tr>
<tr>
<td>( a_m )</td>
<td>( 1.024 \times 10^{-2} ) ( m^2 )</td>
<td></td>
</tr>
<tr>
<td>( N )</td>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>( \eta_m )</td>
<td>( 4 \times 10^{-7} ) H/m</td>
<td></td>
</tr>
<tr>
<td>( \delta_{ref} )</td>
<td>0.004 m</td>
<td></td>
</tr>
<tr>
<td>( \delta_{max} )</td>
<td>( -0.001 ) m</td>
<td></td>
</tr>
<tr>
<td>( \delta_{min} )</td>
<td>0.001 m</td>
<td></td>
</tr>
<tr>
<td>( \delta_{max} )</td>
<td>( -1 ) A</td>
<td></td>
</tr>
<tr>
<td>( \delta_{min} )</td>
<td>1 A</td>
<td></td>
</tr>
</tbody>
</table>

Table 5
Uncertain parameters of the EMS.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Uncertainty (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>150 kg</td>
<td>( \pm 33 )</td>
</tr>
<tr>
<td>( R_m )</td>
<td>1.1 ( \Omega )</td>
<td>( \pm 50 )</td>
</tr>
<tr>
<td>( a_m )</td>
<td>( 1.024 \times 10^{-2} ) ( m^2 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( N )</td>
<td>1000</td>
<td>0</td>
</tr>
</tbody>
</table>

Since \( M_{ij} \) are normalized, therefore, membership functions are obtained as \( h_i = \prod_{j=1}^{3} M_{ij} \).

B. Simulation of EMS MAGLEV Train

In this section, Theorem 1 is applied to the equivalent TS fuzzy
system of the EMS system and gain feedback matrices are designed. The parameter values of the TS fuzzy system are given in Table 4. By letting $\varepsilon = 0.9$, $\rho = 200$, $\alpha = 10^{-3}$, $\rho = 1$ and $\omega_m = 10$ in Theorem 1, one will have the $L_1$ performance $\Omega = 0.0446$ and

$$
P = 10^{-3} \begin{bmatrix}
0.4462 & 0.0100 & 0.0989 \\
0.0100 & 0.0121 & -0.0162 \\
0.0989 & -0.0162 & 0.0578
\end{bmatrix}
$$

$F_1 = [-0.0142 -0.0229 -0.0076];$

$F_2 = [-0.0046 -0.0023 -0.0054];$

$F_3 = [-0.0458 -0.0033 -0.0139];$

$F_3 = [-0.0184 -0.0052 -0.0040];$

**Case 1.** In this stage, the proposed approach is applied to the nonlinear dynamic of the EMS system (46). The initial state of the EMS system is chosen as $x(0) = [0.0001 0 0]^T$. In addition, the disturbance input is chosen as constant $v = 45$. Figs. 4 and 5 demonstrate the state evolution of closed-loop nonlinear system and control effort that is exerted to the system. It is clear that the proposed saturated controller in this paper successfully attenuates the effect of the persistent bounded disturbance signal on the system output.

**Case 2.** In this case, to more challenge the proposed method and in order to show the robustness of the proposed controller against changes in system parameters and un-modeled dynamics, some changes are made in several system parameters and a noise is added to the nonlinear system. In practice, the EMS system is characterized by uncertainties regarding to temperature change and mass change (Michail et al., 2014). The percentages of maximum variations of the uncertainties are presented in Table 5.

As it can be seen in Table 5, two main sources of parameter uncertainties exist (Michail et al., 2014): Gross mass of carriage and electromagnet and coil resistance. These parameters are simulated by $m(t) = 150(1 + 0.33\sin(2t))$ and $R_c(t) = 1.1(1 + 0.5\cos(3t))$. In addition, the disturbance input is chosen as constant $v = 45$. Applying the control designed in Case 1, provides the state evolution of close-loop nonlinear system and control effort which are demonstrated in Figs. 6 and 7, respectively. As it can be seen in Figs. 6 and 7, the proposed approach can effectively handle the practical parameter uncertainty. However, in this case, the closed-loop system states converge to their equilibrium point slower than Case 2. Furthermore, by comparing the two control signal inputs drawn in Figs. 5 and 7, one concludes that the control input in the Case 2 experiments more oscillatory behavior and is saturated for a longer time.

**5. Conclusion**

In this paper, a systematic approach to $L_1$ performance criterion of saturated input nonlinear systems described by TS fuzzy system is proposed. By utilizing the PDC controller and quadratic Lyapunov function, the robust $L_1$ conditions are formulated in terms of linear matrix inequalities (LMIs) and generalized eigenvalue problem (G EVP). The proposed condition leads to less conservative results compared to the one concerning the $L_1$ performance criterion and actuator saturation limitation. In the viewpoint of $L_1$ performance, the robust controller alleviates the effect of persistent bounded disturbance to the system output. Furthermore, in the formulating procedure, some new scalars are introduced not only increase the degree of freedom but also, leads to less number of pre-fixed parameters in the LMI conditions. The outcome is achieving less $L_1$ performance gain. In the viewpoint of actuator saturation, by utilizing the S-procedure, more relaxed LMI conditions are derived. Consequently, these conditions can be employed to wider classes of TS systems and provide a larger region of attraction. Different numerical examples are presented to verify these claims. The comparison results show the advantages of the proposed techniques in the case of both aforementioned viewpoints.

It should be noted that the consideration of $L_1$ performance criterion together with input saturation is not studied to be knowledge of the authors. However, in this paper, this issue is addressed and the proposed conditions are employed to guarantee the stability of the closed-loop nonlinear EMS system with an attenuation $L_1$ performance level and actuator saturation constraint. The achieved results confirm the applicability and efficiency of the proposed approach for real nonlinear processes.

**References**


