Design of $H_\infty$ robust fault detection filter for linear uncertain time-delay systems

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Abstract

In this paper, the robust fault detection filter design problem for linear time-delay systems with both unknown inputs and parameter uncertainties is studied. Using a multiobjective optimization technique, a new performance index is introduced, which takes into account the robustness of the fault detection filter against disturbances and sensitivity to faults simultaneously. The reference residual model is then designed based on this performance index to formulate the robust fault detection filter design problem as an $H_\infty$ model-matching problem. By applying robust $H_\infty$ optimization control technique, the existence condition of the robust fault detection filter for linear time-delay systems with both unknown inputs and parameter uncertainties is presented in terms of linear matrix inequality formulation, independently of time delay. In order to detect the fault, an adaptive threshold which depends on the inputs is finally determined. An illustrative design example is used to demonstrate the validity of the proposed approach. © 2006 ISA—The Instrumentation, Systems, and Automation Society.

Keywords: Uncertain time-delay systems; Robust fault detection filter; $H_\infty$ optimization; LMI

1. Introduction

Due to an increasing demand for higher performance, as well as due to higher safety and reliability standards, the model-based approaches to fault detection and isolation (FDI) in automatic processes have received a great deal of attention during last two decades. Among the model-based approaches, the most common way is the observer-based approach [1], i.e., using state observers or filters to generate residuals and using these residuals to set a threshold to detect the fault. Recently, with the rapid development of robust control theory and $H_\infty$ optimization control technique, more and more methods have been presented to solve robust fault detection and isolation (RFDI) problems, see for example Refs. [1–5], and references therein. Different from robust control, the goal of robust fault detection is to discriminate between the fault effects and the effects of uncertain signals and perturbations. Therefore, the performance of RFDI systems should be measured by a suitable trade-off between robustness and sensitivity.

As is well known, the dynamic behavior of many physical processes contains, more or less, time delays and uncertainties, and can be modeled by an uncertain system with state delay [6]. Since the delayed state is very often the cause for instability and poor performance of systems, increasing attention has recently been devoted to the robust fault detection filter (RFDF) design problems of the linear uncertain state delayed systems, see for example Refs. [7,8]. In Ref. [7], the RFDF problems for time-delay LTI systems were studied via introducing an ideal reference residual model. The method presented in Ref. [7] requires the matrix $D_f$ is of full column rank; in fact, when consider-
ing the faults appearing in state and output equations, it is impossible for the matrix $D$ to have full column rank. In Ref. [8], by using the $H_{\infty}$ control theory, a robust fault detection scheme was proposed for a class of discrete time-delay systems with uncertainty. However, it did not consider the sensitivity of the residual signals to the faults. In addition, the existing results achieved in delay-free systems are not suitable for time-delay systems because of the existence of state delay. For instance, a linear matrix inequality (LMI) approach to design RFDF for uncertain LTI systems was studied in Ref. [5]. However, the performance index used to design the reference residual model cannot be extended to time-delay systems due to the existence of time delay in state. So far, little attention has been paid to the RFDF design problems in the simultaneous presence of time delay, exogenous disturbance inputs, and parametric uncertainties. This motivates the present research on designing an $H_{\infty}$ RFDF for linear time-delay systems with both exogenous disturbance inputs and norm-bounded uncertainties.

This paper deals with the problem of an $H_{\infty}$ RFDF design for a class of linear uncertain systems with a delayed state and exogenous disturbances. This problem aims at designing the RFDF such that, for both exogenous disturbances and admissible parameter uncertainties, the RFDF system is stable and the $H_{\infty}$ norm of the system is less than a prescribed level, independently of the time delay. The time delay is assumed to be unknown, and the parameter uncertainties are allowed to be norm-bounded and may be time varying. In this paper, a reference residual model is used to reduce the RFDF design problem to a standard $H_{\infty}$ model-matching problem. The reference residual model is designed based on a new preference index, which takes into account the robustness against disturbances and sensitivity to faults simultaneously. In addition, applying the $H_{\infty}$ optimization technique, an independent-delay LMI approach to design the RFDF and an adaptive threshold are also proposed in this paper.

The rest of the paper is arranged as follows. In Section 2, the RFDF design problem is formulated for linear uncertain time-delay systems. The main results as well as detail derivations are given in Section 3, including the choice of reference residual model, the design of a RFDF, and the design of adaptive threshold. We demonstrate the validity of the proposed method in Section 4 by an illustrative design example, including abrupt fault case and incipient fault case. Finally, a conclusion is given in Section 5 to end this paper.

2. Preliminaries and problem formulation

Throughout this paper, the superscript “T” stands for matrix transposition, $R^n$ denotes the $n$ dimensional Euclidean space, $R^{n \times m}$ is the set of all real matrices, and the notation $P > 0 (<0)$ for $P \in R^{n \times n}$ means that $P$ is symmetric and positive-definite (negative-definite). The space of functions in $R^q$ that are square integrable over $[0, \infty)$ is denoted by $L_2^q[0, \infty)$. $I$ denotes identity matrix with appropriate dimensions. All matrices, if their dimensions are not explicitly stated, are assumed to be compatible. For a vector function $h(t) \in L_2^q[0, \infty)$, we define its norm $\|h(t)\|_2 = [\int_0^\infty h^T(t)h(t)dt]^{1/2}$. The $H_{\infty}$ norm of a transfer function $G(s)$ is denoted by $\|G(s)\|_{\infty} = \sup_{s \in C} \sigma_{\text{max}}[G(jwo)]$, here $\sigma_{\text{max}}(\cdot)$ denotes the largest singular value of $(\cdot)$. $\star$ denotes the elements below the main diagonal of a symmetric block matrix.

Consider the following fault uncertain systems with a single state-delay

\[
\dot{x}(t) = (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - \tau) + (B + \Delta B)u(t) + B_d f(t) + B_dd(t),
\]

\[
y(t) = Cx(t) + Du(t) + D_f f(t) + D_dd(t),
\]

\[
x(t) = 0 \quad \forall \ t \in [-h, 0],
\]

where $x(t) \in R^n$ is the state vector, $u(t) \in R^p$ is the control input vector, $y(t) \in R^q$ is the measurement output vector, $d(t) \in R^m$ is the disturbance input that belongs to $L_2^m[0, +\infty)$, $f(t) \in R^l$ is the fault to be detected, $h$ is an unknown but constant delay. $A, A_d, B, B_d, C, D, D_f,$ and $D_dd$ are known matrices with appropriate dimensions. $\Delta A, \Delta A_d, \Delta B$ are real-valued matrix functions representing norm-bounded parameter uncertainties and satisfy

\[
[\Delta A \ \Delta A_d \ \Delta B] = [E_1 \Sigma_1 F_1 \ E_2 \Sigma_2 F_2 \ E_3 \Sigma_3 F_3],
\]

where $E_1, E_2, E_3, F_1, F_2,$ and $F_3$ are known real constant matrices of appropriate dimensions which specify how the uncertain parameters in $\Sigma_1, \Sigma_2,$
and $\Sigma_3$ enter the nominal matrices $A$, $A_d$ and $B$; $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$, which may be time-varying, are real unknown matrices with Lebesgue measurable elements and satisfy $\Sigma_1^T \Sigma_1 \leq I$, $\Sigma_2^T \Sigma_2 \leq I$, and $\Sigma_3^T \Sigma_3 \leq I$. Denote

$$\begin{align*}
\Omega_1 &= \{ \Delta A | \Delta A = E_1 \Sigma_1 F_1, \Sigma_1^T \Sigma_1 \leq I \} \\
\Omega_2 &= \{ \Delta A_d | \Delta A_d = E_2 \Sigma_2 F_2, \Sigma_2^T \Sigma_2 \leq I \} \\
\Omega_3 &= \{ \Delta B | \Delta B = E_3 \Sigma_3 F_3, \Sigma_3^T \Sigma_3 \leq I \}.
\end{align*}$$

(4)

We are interested in designing the so-called fault detection filter

$$\dot{\hat{x}}(t) = A \hat{x}(t) + A_d \hat{x}(t-h) + Bu(t) + H[y(t) - \hat{y}(t)],$$

(5)

$$\hat{y}(t) = C \hat{x}(t) + Du(t), \quad r(t) = V[y(t) - \hat{y}(t)],$$

(6)

where $\hat{x}(t) \in R^n$ and $\hat{y}(t) \in R^m$ represent the state and output estimation vectors, respectively; $r(t)$ is the so-called residual signal. The design parameters of a RFDF are the observer gain matrix $H$ and the residual weighting matrix $V$. Define the error state $e(t) = x(t) - \hat{x}(t)$. Then it follows from (1), (2), (5), and (6) that

$$\dot{e}(t) = (A - HC)e(t) + A_d e(t-h) + \Delta A x(t)$$

$$+ \Delta A_d x(t-h) + \Delta B u(t) + (B_f - HD_f)f(t)$$

$$+ (B_d - HD_d)d(t),$$

(7)

$$r(t) = V C e(t) + V D_f f(t) + V D_d d(t).$$

(8)

Note that the dynamics of the residual signal in (8) depends not only on $f(t)$, $d(t)$, and $u(t)$, but also on the states $x(t)$ and $x(t-h)$. The methods mentioned in Refs. [1–5] for delay-free cases are not suitable to solve the RDFD design problem, because there are time delays in $e(t-h)$ and $x(t-h)$. Here, we propose to use a reference residual model, describing the desired behavior of the residual vector $r(t)$, to formulate the RDFD design problem as an $H_\infty$ model-matching problem, that is to find an idealized reference residual and minimize the worst case distance between the generated residual and the idealized reference residual. We suppose that the reference residual model (that is, $\Delta A=0$, $\Delta A_d=0$, $\Delta B=0$) is given by

$$r_f(s) = W_f(s)f(s) + W_d d(s),$$

(9)

where $W_f(s) \in RH_\infty$, $W_d(s) \in RH_\infty$. According to Ref. [7] and assuming $\Delta A=0$, $\Delta A_d=0$, $\Delta B=0$, the corresponding reference residual error state model is given by

$$\dot{e}_f(t) = (A - H^* C)e_f(t) + A_d e_f(t-h) + (B_f - H^* D_f)f(t) - H^* D_d d(t),$$

(10)

$$r_f(t) = V^* C e_f(t) + V^* D_f f(t) + V^* D_d d(t),$$

(11)

where $e_f(t) \in R^n$ is the reference error state vector, $r_f(t)$ is the reference residual signal, $H^*$ and $V^*$ are the parameters of the reference model to be designed.

Thus the overall system can be described by

$$\dot{\bar{x}}(t) = (\bar{A} + \Delta \bar{A}) \bar{x}(t) + (\bar{A}_d + \Delta \bar{A}_d) \bar{x}(t-h) + (\bar{B} + \Delta \bar{B}) w(t),$$

(12)

$$r_e(t) := r(t) - r_f(t) = \bar{C} \bar{x}(t) + \bar{D} w(t),$$

(13)

where

$$\bar{e} = \begin{bmatrix} e \\ e_f \\ x \end{bmatrix}, \quad w = \begin{bmatrix} u \\ f \\ d \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} A - HC & 0 & 0 \\ 0 & A - H^* C & 0 \\ 0 & 0 & A \end{bmatrix},$$

$$\bar{A}_d = \begin{bmatrix} A_d & 0 & 0 \\ 0 & A_d & 0 \\ 0 & 0 & A_d \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} B_f - H D_f & B_d - H D_d \\ 0 & B_f - H^* D_f & B_d - H^* D_d \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} V C & -V^* C & 0 \end{bmatrix},$$

$$\bar{D} = \begin{bmatrix} 0 & V D_f & V D_d - V^* D_d \end{bmatrix},$$

$$\Delta \bar{A} = \bar{E}_1 \bar{\Sigma}_1 \bar{F}_1,$$

$$\Delta \bar{A}_d = \bar{E}_2 \bar{\Sigma}_2 \bar{F}_2,$$

$$\Delta \bar{B} = \bar{E}_3 \bar{\Sigma}_3 \bar{F}_3,$$
\[
\vec{E}_1 = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad \vec{E}_2 = \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, \quad \vec{E}_3 = \begin{bmatrix} E_3 \end{bmatrix},
\]

\[
\vec{F}_1 = [0 \ 0 \ F_1], \quad \vec{F}_2 = [0 \ 0 \ F_2], \quad \vec{F}_3 = [F_3 \ 0 \ 0].
\]

Here we can see that the design of robust fault detection filter problem, the one of main objectives of this work, is formulated as an \(H_\infty\) model-matching design problem, i.e., applying robust \(H_\infty\) optimization control technique, for all admissible parameter uncertainties, the system residual \(r(t)\) will be designed as closely as possible to the reference model residual \(r_f(t)\), independently of the unknown time-delay \(h\). Thus the problem of designing an observer-based RFDF can be described as designing observer gain matrix \(H\) and residual weighting matrix \(V\) such that:

1. System (12) and (13) is robustly asymptotically stable; and
2. The following performance index \(J\), under zero initial condition, is made small

\[
J = \sup_{w \in L_2, w \neq 0} \left\| \frac{r(t)}{w(t)} \right\|_2.
\]

After designing of a RFDF, the remaining important task of this paper is the evaluation of the generated residual. One of the widely adopted approaches is to choose a so-called threshold \(J_{th} > 0\) and, based on this, use the following logical relationship for fault detection:

\[
\begin{align*}
\|r(t)\|_{2,\tau} &> J_{th} \Rightarrow \text{a fault has occurred} \Rightarrow \text{alarm} \\
\|r(t)\|_{2,\tau} &\leq J_{th} \Rightarrow \text{no fault has occurred},
\end{align*}
\]

where \(\|r(t)\|_{2,\tau} = \left[\int_{t_1}^{t_2} r(t)^T r(t) dt\right]^{1/2}, \quad \tau = t_1 - t_2, \quad t \in [t_1, t_2].\)

Note that the length of the time window is finite (i.e., \(\tau\) instead of \(\infty\)). Since in practice it is desired that the faults will be detected as early as possible, an evaluation of residual signal over the whole time range makes less sense. This point has been mentioned in Refs. [5, 7].

3. Main results and proofs

As mentioned early, the design of a RFDF for linear uncertain time-delay systems can be formulated as an \(H_\infty\) model-matching problem. We should design the reference residual model first; the RFDF design problem for linear uncertain time-delay systems can then be solved. In addition, we note that the residual signal generated in (8) depends on the input \(u(t)\), therefore the threshold should be designed adaptively according to the changes of \(u(t)\). So in this section, there are three problems to be solved, that is (a) choice of reference residual model, (b) design of RFDF for linear uncertain time-delay system in terms of the LMI formulation, and (c) design of the adaptive threshold.

3.1. Choice of reference residual model

The selection of a suitable reference model is one of key steps to design an RFDF for linear uncertain time-delay systems. If the reference residual model is not selected suitably, the reference residual \(r_f(t)\) cannot describe the desired behavior of the generated residual \(r(t)\). As a result, more misses or false alarms may be occurring. Therefore, the reference residual model should be designed such that:

1. The influence of the exogenous disturbances to the reference residual \(\rightarrow \text{min}\); and
2. The effect of the faults, i.e., the fault sensitivity to the reference residual \(\rightarrow \text{max}\).

In fact, the mathematical interpretation of the requirement on the high sensitivity to the faults and simultaneously the strong robustness to the exogenous disturbance inputs is a multiobjective optimization control problem. In order to select a suitable trade-off, we first consider the following performance index:

\[
J_f = \|G_{rf}\|_\infty - \|G_{r_d}\|_\infty
\]

where \(G_{rf}\) and \(G_{r_d}\) are the transfer functions from \(f\) and \(d\) to reference residual \(r_f\), respectively.

It is interesting to notice that if setting \(J_f \rightarrow \text{max}\), we have

\[
\|G_{rf}\|_\infty \rightarrow \text{max} \quad \text{and} \quad \|G_{r_d}\|_\infty \rightarrow \text{min}.
\]

Therefore, the reference residual model (10) and (11) can be designed according to the performance index (16), which takes into account the robustness of the reference residual against disturbances and sensitivity to faults simultaneously.

For the sake of simplicity, we assume that \(l = m\). In fact, if \(l > m\) (or \(l < m\)), by extending \(G_{r_d}\)
(or $G_{rf}$) and $d$ (or $f$) to $\tilde{G}_{rf,d} = [G_{rf,d} \phi_{m-1}]$ (or $\tilde{G}_{rf,f} = [G_{rf,f} \phi_{m-1}]$) and $\tilde{d} = [d \phi_{m-1}]$ (or $\tilde{f} = [f \phi_{m-1}]$), we can have the same results, where $\phi$ denotes null matrix with appropriate dimensions.

Here we introduce two matrices $L$ and $R$ to select the appropriate input/output channels or channel combinations, which was used in multiobjective output-feedback control in Ref. [9]. Consider the following transfer function:

$$G \triangleq LG_{rf,d}R = L[G_{rf,f} G_{rf,d}]R,$$  \hspace{1cm} (18)

where $L \in R^{q \times q}$ and $R \in R^{l \times l}$. For some prescribed $\beta > \alpha > 0$, if choosing $L = I_{q \times q}$ and $R = [I_{l \times l}]$, then we have

$$\|G\|_\infty = \|G_{rf,f} - G_{rf,d}\|_\infty \geq \|G_{rf,f}\|_\infty - \|G_{rf,d}\|_\infty,$$

$$\beta - \alpha \Rightarrow \|G_{rf,f}\|_\infty > \beta \text{ and } \|G_{rf,d}\|_\infty < \alpha.$$  \hspace{1cm} (19)

$G$ admits the following realization:

$$\begin{bmatrix}
-A^T P - PA + C^T Y^T + Y C + C^T Z C - Q & P(B_d - B_f) + Y(D_f - D_d) + C^T Z (D_f - D_d) - PA_d \\
\ast & (D_f - D_d)^T Z (D_f - D_d) - (\beta - \alpha)^2 I \ast \ast Q
\end{bmatrix} > 0$$  \hspace{1cm} (22)

holds, then the reference residual model (10) and (11) is stable and (21) holds, moreover $H^\infty = P^{-1} Y$, $V^\infty = Z^{1/2}$, where $Z^{1/2}$ denotes the square root factorization of the matrix $Z$.

**Proof** See Appendix A.

**Remark 1** In many cases without modeling errors, the generated residual $r(t)$ is not totally decoupled from the unknown input $d(t)$. Therefore, we choose (9) as the reference residual model.

**Remark 2** For (18), if we choose $L = I_{q \times q}$ and $R = [I_{l \times l}]$, the optimization problem (21) is the sensitivity of the reference residual to the faults; if we choose $L = I_{q \times q}$ and $R = [\phi_{m-1}]$, the problem (21) is the robustness of the reference residual against the disturbance.

### 3.2. Design of a RFDF

The next thing to be solved is to design the RFDF for linear uncertain time-delay systems, which is formulated as an $H^\infty$ model-matching problem and solved via a LMI formulation. The following theorem presents a sufficient condition to guarantee that not only the system is asymptotically stable but also the $H^\infty$-norm of the system is less than a prescribed level, independently of time delay.

**Theorem 2** Consider the linear uncertain time-delay system
\[ x(t) = (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - h) + (B + \Delta B)w(t), \]
\[ z(t) = Cx(t) + Dw(t), \quad x(t) = 0 \quad \forall t \in [-h, 0] \]

(23)

(24)

holds, then for \( \Delta A \in \Omega_1, \Delta A_d \in \Omega_2, \Delta B \in \Omega_3 \) the system (23) and (24) is robustly asymptotically stable and satisfies \( \|z\|_2 \leq \gamma\|w\|_2 \).

**Proof** See Appendix B.

Using Theorem 2, the RFDF design problem formulated earlier can be easily solved.

**Theorem 3** For a given positive constant \( \gamma > 0 \), if there exist symmetric positive-definite matrices \( P_1 > 0, P_2 > 0, P_3 > 0, Q_1 > 0, Q_2 > 0, Q_3 > 0 \) and \( \Lambda = \text{diag}(\epsilon_1 I, \epsilon_2 I, \epsilon_3 I) > 0 \) such as matrices \( Y_1 \) and \( V \) such that LMI

\[
\begin{bmatrix}
PA + A^TP + Q + \epsilon_1^{-1}F_1^TF_1 & PB & C^T \\
* & -\gamma^2 I + \epsilon_2^{-1}F_3^TF_3 & D^T \\
* & * & -I \\
* & * & -Q + \epsilon_2^{-1}F_2^TF_2 \\
* & * & * & -\epsilon_1^{-1}I \\
* & * & * & * & -\epsilon_2^{-1}I \\
* & * & * & * & * & -\epsilon_3^{-1}I
\end{bmatrix} < 0
\]

(25)

holds, then the system (10) and (11) is robustly asymptotically stable and satisfies \( \|r\|_2 \leq \gamma\|w\|_2 \). Furthermore, the observer gain matrix is given by

\[
H = P_1^{-1}Y_1,
\]

(26)

where \( N_{0101} = P_1A + A^TP + Q_1 - Y_1C - C^TY_1^T \), \( N_{0105} = P_1B - Y_1D_f \), \( N_{0106} = P_1B_2 - Y_1D_d \), \( N_{0107} = C^TY_1^T \), \( N_{0108} = P_1A_d \), \( N_{0111} = P_1E_1 \), \( N_{0112} = P_1E_2 \), \( N_{0113} = P_1E_3 \), \( N_{0202} = P_2A + A^TP_2 + Q_2 - P_2H^TC - C^TH^TP_2 \), \( N_{0205} = P_2B_2 - P_2H^TD_f \), \( N_{0206} = P_2B_d \), \( N_{0207} = -C^TY_2^T \), \( N_{0209} = P_2A_d \), \( N_{0303} = P_3A + A^TP_3 + Q_3 + \epsilon_1^{-1}F_3^TF_3 \), \( N_{0304} = P_3B \), \( N_{0305} = P_3B_f \), \( N_{0306} = P_3B_d \), \( N_{0310} = P_3A_d \), \( N_{0311} = P_3E_1 \), \( N_{0312} = P_3E_2 \), \( N_{0313} = P_3E_3 \), \( N_{0404} = -\gamma^2 I + \epsilon_3^{-1}F_3^TF_3 \), \( N_{0505} = -\gamma^2 I \), \( N_{0507} = D_1^TV^T - D_1^TV^T \), \( N_{0606} = -\gamma^2 I \), \( N_{0607} = D_2^TV^T - D_2^TV^T \), \( N_{0707} = -I \), \( N_{0808} = -Q_1 \), \( N_{0909} = -Q_2 \), \( N_{1010} = -Q_3 + \epsilon_1^{-1}F_2^TF_2 \), \( N_{1111} = -\epsilon_1^{-1}I \), \( N_{1212} = -\epsilon_2^{-1}I \), \( N_{1313} = -\epsilon_3^{-1}I \), \( N_{ij} = 0 \) otherwise.

**Proof** See Appendix C.

The condition (26) is a LMI condition. Therefore, for fixed \( \gamma > 0 \), the observer gain matrix \( H \) and the residual weighting matrix \( V \) can be easily obtained by using some software such as Matlab LMI toolbox.

### 3.3. Design of adaptive threshold

The last step of fault detection is to evaluate the residual. This is a decision-making process that always comes down to the threshold logic of a decision function. The method used in this paper is similar to the one in Ref. [5]. Considering the residual (8), the fault-free case residual evaluation function is

\[
\|r_d(t) + r_w(t)\|_{2, r} \leq \|r_d(t)\|_{2, r} + \|r_w(t)\|_{2, r} \leq J_d + J_w,
\]

(28)
We can see that if the input changes, the threshold $J_{th}$ is also changed with it. Therefore, the threshold $J_{th}$ can be determined by

$$J_{th} = M + J_u.$$  

(30)

Note that the defined threshold consists of two parts: the constant part $M$ and the variable part $J_u$. The part $J_u$ depends on the input $u$ and can be calculated on-line. We can see that if the input $u(t)$ changes, the threshold $J_{th}$ is also changed with it. So the threshold is adaptive in this sense.

### 4. Numerical example and simulation

In this section, we demonstrate the theory developed in this paper by means of an example. Consider the linear uncertain time-delay system (1) and (2) with parameters given by

$$A = \begin{bmatrix} 3.8 & 1.5 & -0.5 \\ 0.5 & -3.0 & 1.0 \\ -0.3 & 0.7 & -2.4 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0.4 & 0.1 & -0.2 \\ 0.1 & -0.8 & 0.2 \\ 0.7 & -0.1 & 0.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.4 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0.6 & 0 \\ -0.5 & 0 \\ 0.4 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D = 0, \quad D_j = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.8 \\ 0 & -1.2 \end{bmatrix},$$

$$D_j = \begin{bmatrix} 0.9 & 0 \\ 0.2 & 0 \\ 0.7 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.1 \\ 0 \\ -0.1 \end{bmatrix},$$

where $r_d(t) := r(t)_{\Delta t = 0, \Delta \\ d = 0, f = 0}$, $r_u(t) := r(t)_{\Delta t = 0, \Delta \\ d = 0, f = 0}$, $J_d = \sup_{\Delta A \in \Omega_1, \Delta A_d \in \Omega_2} \sup_{\Delta B \in \Omega_1, \Delta d = 0} \| r_d \|_{2, r}$, $J_u = \sup_{\Delta A \in \Omega_1, \Delta A_d \in \Omega_2} \| r_u \|_{2, r}$.

So we choose the threshold $J_{th}$ as

$$J_{th} = J_d + J_u.$$  

(29)

Under the assumption of $\Delta d \in L_2$, we can further have $\sup_{\Delta A \in \Omega_1, \Delta A_d \in \Omega_2} \sup_{\Delta B \in \Omega_1, \Delta d = 0} \| r_d \|_{2, r} = M$ $(M > 0)$. By using Theorem 2, we can obtain $J_u = \gamma_a \| u \|_{2, r}$, where $\gamma_a = \sup_{\Delta A \in \Omega_1, \Delta A_d \in \Omega_2} \| r_u \|_{2, r}/\| u \|_{2, r}$. Therefore, the threshold $J_{th}$ can be determined by

$$J_{th} = M + \gamma_a \| u \|_{2, r}.$$  

(30)

Using the Matlab LMI Control Toolbox to solve the LMI (26), we can obtain the solution as

$$H^* = \begin{bmatrix} -0.4563 & -1.2761 & 0.0935 \\ 0.5954 & -0.5642 & 0.1101 \\ -0.1407 & 1.7430 & -0.8888 \end{bmatrix},$$

$$V^* = \begin{bmatrix} 0.9636 & -0.1493 & 0.0028 \\ 0 & 0.7926 & 0.1775 \\ 0 & 0 & 0.8146 \end{bmatrix},$$

$$\beta_{max} = 0.9980.$$  

Using the Matlab LMI Control Toolbox to solve the LMI (26), we can obtain the solution as

$$P_1 = \begin{bmatrix} 0.6054 & 0.5623 & 0.0992 \\ 0.5623 & 2.8216 & 0.5712 \\ 0.0992 & 0.5712 & 3.4514 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.3718 & 0.0841 & 0.0937 \\ 0.0841 & 0.7519 & 0.0450 \\ 0.0937 & 0.0450 & 0.5383 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.0050 & 0.0001 & 0.0041 \\ 0.0001 & 0.0185 & 0.0144 \\ 0.0041 & 0.0144 & 0.0279 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1.4765 & -0.2809 & 1.0893 \\ -0.2809 & 2.3407 & -1.0319 \\ 1.0893 & -1.0319 & 1.4602 \end{bmatrix}.$$
\[ Q_2 = \begin{bmatrix} 0.5261 & -0.1111 & 0.3459 \\ -0.1111 & 0.6232 & -0.0107 \\ 0.3459 & -0.0107 & 0.3461 \end{bmatrix}, \]
\[ Q_3 = \begin{bmatrix} 0.0198 & -0.0104 & 0.0140 \\ -0.0104 & 0.0144 & -0.0090 \\ 0.0140 & -0.0090 & 0.0125 \end{bmatrix}, \]
\[ Y_1 = \begin{bmatrix} 1.0594 & 4.2153 & -2.4499 \\ -0.3942 & 4.1267 & -1.8940 \\ -0.5943 & -7.9327 & 4.6913 \end{bmatrix}, \]
\[ V = \begin{bmatrix} 1.2546 & 0.7388 & -0.3177 \\ 0.1219 & 1.9965 & -0.5061 \\ -0.1078 & -1.8516 & 1.9697 \end{bmatrix}, \]
\[ \varepsilon_1 = 2.7661, \quad \varepsilon_2 = 1.0014, \]
\[ \varepsilon_3 = 0.0552, \quad \gamma_{\text{min}} = 0.4862. \]

Therefore, by Theorem 3 the robust fault detection filter problem is solvable, and a desired filter is given by (5) and (6) with
\[
H = \begin{bmatrix} 2.3021 & 6.8000 & -4.1559 \\ -0.5693 & 0.6336 & -0.1474 \\ -0.1442 & -2.5988 & 1.5031 \end{bmatrix}, \]
\[
V = \begin{bmatrix} 1.2546 & 0.7388 & -0.3177 \\ 0.1219 & 1.9965 & -0.5061 \\ -0.1078 & -1.8516 & 1.9697 \end{bmatrix}. \]

In real physical systems, the fault is developing either quickly or slowly; therefore we consider the abrupt (sudden) fault case and the incipient (slowly developing) fault case, respectively, to demonstrate the effectiveness of the design. The time delay \( h \) is assumed to be 1 s, an unknown input \( d(t) \) is assumed to be band-limited white noise with power 0.005 (zero-order holds with sampling time 0.01 s) and the input \( u(t) \) is taken as unit step signal.

### 4.1. Abrupt fault case

The fault signal \( f(t) \) is simulated as a square wave of unite amplitude occurred from 5 to 10 s. The fault signal \( f(t) \) and the generated residual signal \( r(t) \) [including \( r_1(t) \), \( r_2(t) \) and \( r_3(t) \)] are shown in Figs. 1 and 2, respectively. Fig. 3 shows the evolution of residual evaluation function \( \|r(t)\|_{2, \gamma} \), from which we can calculate when the fault can be detected. By using Theorem 2, we have minimal \( \gamma_u = 0.3867 \). Suppose \( M = 0.5 \), and then the threshold is
\[
J_{\text{th}} = M + \gamma_u\|u\|_{2,6.3} = 0.5 + 0.3867 \times \left[ \int_0^{6.3} u^T(t)u(t) \, dt \right]^{1/2} = 1.4706. \]

In Fig. 3, we can see that \( \|r(t)\|_{2,6.3} = 1.5 > 1.4706 \) for \( t_1 = 0 \) s and \( t_2 = 6.3 \) s. This means that the fault \( f(t) \) can be detected 1.3 s after its occurrence.
4.2. Incipient fault case

In this case, the fault \( f(t) \) is assumed as an incipient fault, which is shown in Fig. 4. The generated residual signals \( r(t) \) [including \( r_1(t) \), \( r_2(t) \), and \( r_3(t) \)] and the evolution of residual evaluation function \( \|r(t)\|_{2,r} \) are illustrated in Figs. 5 and Fig. 6, respectively. In Fig. 6, we can see that \( \|r(t)\|_{2,6.3} \approx 1.0 < 1.4706 \) for \( t_1 = 0 \) s and \( t_2 = 6.3 \) s. This is because the fault in this case develops more slowly than the one in the above case. Similar to the above case, we can see in Fig. 6 that \( \|r(t)\|_{2,7.8} \approx 1.60 > 1.5799 \) for \( t_1 = 0 \) s and \( t_2 = 7.8 \) s. This means that the fault \( f(t) \) can be detected 2.8 s after its occurrence.

5. Conclusion

In this paper, the observer-based RFDF design problem is studied for linear uncertain time-delay system with both unknown inputs and parameter uncertainties. The main contribution of our study is the introduction of a new performance index, which takes into account the robustness of the fault detection filter against disturbances and sensitivity to faults simultaneously. Based on this performance index, an optimal reference residual model (considering both observer gain matrix and residual weighting matrix) has been designed to formulate the RFDF design problem as an \( H_\infty \) model-matching problem. The existence of the solution is presented in terms of the LMI formulation, which can be obtained conveniently by using Matlab LMI toolbox. An illustrative example has demonstrated the validity of the proposed approach.
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Appendix A. Proof of Theorem 1
Define the Lyapunov-Krasovskii function candidate as follows:

\[
V_1(\epsilon(t), t) = \epsilon^T(t)P\epsilon(t) + \int_{t-h}^{t} \epsilon^T(\theta)Q\epsilon(\theta) d\theta,
\]

(A1)

where \(P > 0\) and \(Q > 0\).

Consider the following index:

\[
J_1 = \int_{0}^{\infty} r_f^T r_f dt - (\beta - \alpha)^2 \int_{0}^{\infty} \dot{\vartheta}^T \dot{\vartheta} dt
\]

\[
= \int_{0}^{\infty} \left[ \epsilon(t) \dot{\vartheta}(t) - (\beta - \alpha)^2 \dot{\vartheta}^T \dot{\vartheta} - \dot{V}(\epsilon(t), t) \right] dt
\]

\[
+ V_1(\epsilon(t), t)|_{t=\infty} - V_1(\epsilon(t), t)|_{t=0}. \quad (A2)
\]

Denote \(Y = PH\), \(Z = \dot{V}\), and choose \(L = I_{q\times q}\), \(R = \begin{bmatrix} I_{q\times q} \\ -I_{q\times q} \end{bmatrix}\). Because \(V_1(\epsilon(t), t)|_{t=0} = 0\) under zero initial condition and \(V_1(\epsilon(t), t)|_{t=\infty} = 0\), substitute the time derivative of \(V_1(\epsilon(t), t)\) along (20) into \(J_1\), then we lead to

\[
J_1 \geq \int_{0}^{\infty} \left[ \epsilon(t) \dot{\vartheta}(t) - (\beta - \alpha)^2 \dot{\vartheta}^T \dot{\vartheta} - \dot{V}(\epsilon(t), t) \right] dt
\]

\[
= \int_{0}^{\infty} \left[ \begin{array}{c} \epsilon(t) \\ \dot{\vartheta}(t) \\ \dot{\vartheta}(t-h) \end{array} \right]^T \Xi \left[ \begin{array}{c} \epsilon(t) \\ \dot{\vartheta}(t) \\ \dot{\vartheta}(t-h) \end{array} \right] dt. \quad (A3)
\]

\(\Xi > 0\) implies that \(J_1 > 0\). According to condition (22), then \(\|G_{\alpha}(s) - G_{\alpha}(s)\|_\infty > \beta - \alpha\) holds. Using (19), then we have \(\|G_{\alpha}\|_\infty > \beta\) and \(\|G_{\alpha}\|_\infty < \alpha\).

Consider the following inequality:

\[
\begin{bmatrix}
A^T P + PA - YC - CT^t Y + Q & PA_d \\
A_d^T P & -Q
\end{bmatrix} < 0,
\]

(A4)

which guarantees \(\dot{V}_1(\epsilon(t), t) < 0\) in case of \(\dot{\vartheta} = 0\). If the LMI (20) is feasible, then the LMI (A4) is also feasible. Thus the system (10) and (11) is stable. This completes the proof. □

Appendix B. Proof of Theorem 2
The following lemmas will be useful in proving Theorem 2.

Lemma 1 [10] Given constant matrices \(X_1, X_2, X_3\), where \(X_1 = X_1^T\) and \(0 < X_2 = X_2^T\), then \(X_1 + X_2 X_3^{-1} X_3 < 0\) if and only if

\[
\begin{bmatrix}
X_1 & X_3^T \\
X_3 & -X_2
\end{bmatrix} < 0 \quad \text{or equivalently} \quad 
\begin{bmatrix}
-X_2 & X_3 \\
X_3^T & X_1
\end{bmatrix} < 0.
\]

Lemma 2 [11,12] Let \(E, F, G, L\), and \(\Sigma\) be real matrices of appropriate dimensions with \(\Sigma = \text{diag}\{\Sigma_1, \Sigma_2, \Sigma_3\}\), \(\Sigma_i \Sigma_i < I\), \(i = 1, 2, 3\). Then for any real matrix \(\Lambda = \text{diag}\{e_1 I, e_2 I, e_3 I\} > 0\), we have

\[
E \Sigma F + E^T \Sigma^T F = E \Lambda E^T + F^T \Lambda^{-1} F.
\]

Let us choose a Lyapunov-Krasovskii function candidate \(V_2[x(t), t]\) as follows:

\[
V_2(x(t), t) = x^T(t)P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta,
\]

where \(P > 0\) and \(Q > 0\).

Consider the following index:

\[
J_2 = \int_{0}^{\infty} [z(t)^T z(t) - \gamma^2 w(t)^T w(t)] dt. \quad (B2)
\]

For the proof of the theorem, it is sufficient to show that \(J_2 < 0\) for any nonzero \(w\) and \(\dot{V}_2(x(t), t) < 0\) for \(w = 0\). The time derivative of \(V_2(x(t), t)\) along the solution of (23) is given by

\[
\dot{V}_2[z(t), t] = [(A + \Delta A)x + (A_d + \Delta A_d)x(t-h)] + (B + \Delta B)w)^T P x + x^T Q x + x^T P [(A + \Delta A)x + (A_d + \Delta A_d)x(t-h)] + (B + \Delta B)w]
\]

\[- x^T(t-h) Q x(t-h). \quad (B3)
\]

We can rewrite \(J_2\) as follows:

\[
J_2 = \int_{0}^{\infty} [z(t)^T z(t) - \gamma^2 w(t)^T w(t)] dt. \quad \text{or} \quad V_2|_{t=0} + V_2|_{t=\infty}.
\]

Because \(V_2|_{t=0} = 0\) under zero initial condition and \(V_2|_{t=\infty} > 0\), we lead to
\[ J_2 \leq \int_0^\infty [z^T z - \gamma^2 w^T w + \dot{V}_2] dt = \int_0^\infty \begin{bmatrix} x & x(t-h) \\ w & w \end{bmatrix}^T \Omega \begin{bmatrix} x & x(t-h) \\ w & w \end{bmatrix} dt, \]

where \( \Omega = \begin{bmatrix} (A + \Delta A)^T P + P(A + \Delta A) + C_1^T C + Q & P(A_d + \Delta A_d) & P(B + \Delta B) + C_1^T D \\ * & -Q & 0 \\ * & * & D^T D - \gamma^2 I \end{bmatrix}. \) (B5)

\( \Omega < 0 \) implies that \( J_2 < 0 \), then we can rewrite \( \Omega < 0 \) as follows:

\[
\Omega_1 + \Omega_2 \times \text{diag}\{\Sigma_1, \Sigma_2, \Sigma_3\} \times \text{diag}\{F_1, F_2, F_3\} + \text{diag}\{F_1^T, F_2^T, F_3^T\} \times \text{diag}\{\Sigma_1, \Sigma_2, \Sigma_3\} \times \Omega_2^T < 0,
\]

where

\[
\Omega_1 = \begin{bmatrix} A^T P + PA + C_1^T C + Q & PA_d & PB + C_1^T D \\ * & -Q & 0 \\ * & * & D^T D - \gamma^2 I \end{bmatrix},
\]

(B6)

and

\[
PA + A^T P + Q + C_1^T C + \varepsilon_1^{-1} F_1^T F_1 \\
* \\
* \\
* \\
PA_d \\
* \\
* \\
* \\
PE_1 \\
* \\
* \\
* \\
PE_2 \\
* \\
* \\
* \\
P + Q + \varepsilon_2^{-1} F_2^T F_2 \\
* \\
* \\
* \\
Q + \varepsilon_1^{-1} I \\
* \\
* \\
* \\
- \varepsilon_1^{-1} I \\
* \\
* \\
* \\
- \varepsilon_2^{-1} I \end{bmatrix} < 0,
\]

(B8)

which guarantees \( \dot{V}_2[x(t), I] \leq 0 \) in case of \( w(t) = 0 \). If the LMI (25) is feasible, then the LMI (B8) is also feasible. Thus the system (23) and (24) is stable. This completes the proof.

Appendix C. Proof of Theorem 3

Define

\[
\begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} > 0, \quad Q = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} > 0
\]

and \( Y_1 = P_1 H \). According to the proof of Theorem 2, the proof of Theorem 3 can be easily obtained. Thus the proof is omitted.

References

[1] Chen, J. and Patton, P. R., Robust Model-Based Fault


