Optimal consumption and portfolio selection problems under loss aversion with downside consumption constraints

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\textbf{A R T I C L E  I N F O}

\textbf{2010 MSC:}
91C10

\textbf{Keywords:}
Loss aversion
Optimal portfolio and consumption
Consumption constraints
Martingale method

\textbf{A B S T R A C T}

This paper investigates continuous-time optimal portfolio and consumption problems under loss aversion in an infinite horizon. The investor’s goal is to choose optimal portfolio and consumption policies to maximize total discounted S-shaped utility from consumption. The consumption rate process is subject to a downside constraint. The optimal consumption and portfolio policies are obtained through the martingale method and replication technique. Numerical results indicate the differences between the loss averse investor and the constant relative risk averse (CRA) investor on the optimal consumption and portfolio policies: the loss averse investor likes consuming more money but exposing less to risk than that of the CRA investor, and the optimal wealth, as a function of state price density, drops faster for the CRA investor than that for the loss averse investor.

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1. Introduction

There have been many research works on financial portfolio selection models since Merton [10] started this research in 1969. The traditional financial portfolio selection models are based on the assumption that investors are rational and risk averse. However, in the real world, substantial experimental evidences are inconsistent with this rationality hypothesis. Kahneman and Tversky [8,17] found that investors treat gains and losses differently, they are risk-averse on gains and risk-seeking on losses, and more sensitive to losses than to gains, and what is more, the investors overweight small probabilities and underweigh large probabilities. Because of these things, Kahneman and Tversky [8] for the first time proposed the concept of loss aversion in the framework of prospect theory, and later Tversky and Kahneman [16] defined it for choice under uncertainty. Loss aversion is an important psychological concept. In mathematics, it can be represented by an utility function which is concave for gains and convex for losses, and steeper for losses than for gains. Loss aversion can explain many phenomena such as the endowment effect (Thaler [15]), the status quo bias (Samuelson and Zeckhauser [12]), and the equity premium puzzle (Benartzi and Thaler [2]), which remain paradoxes in traditional selection theory. Therefore, it has frequently been applied in behavioral finance and received more and more attention. Nevertheless, as far as we know, there are very few literatures available that investigated continuous-time portfolio optimization problems under loss aversion although it has been proposed for ages.


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http://dx.doi.org/10.1016/j.amc.2016.11.029
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portfolio selection model under the cumulated prospect theory (CPT) for the first time, and they developed a systematic approach to overcome the immense difficulties which arise from the analytically ill-behaved utility functions and probability distortions. Zhang, et al. [19] formulated a continuous-time behavioral portfolio selection model where the losses were constrained by a prespecified upper bound and derived the optimal solution explicitly by solving a concave Choquet minimization problem with an additional upper bound. Mi, et al. [11] investigated a dynamic asset allocation model for loss-averse investors in a jump-diffusion financial market, where the optimal wealth process and investment strategies were obtained using martingale approach. Guo [5] studied an optimal portfolio selection problem for the insurer with loss aversion. Fulga [4] presented an approach which incorporates loss aversion preferences in the Mean-Risk framework. However, all of them supposed that the term of investment was finite and the investor was only concerned with the terminal wealth and did not care about the consumption. We investigate the optimal portfolio and consumption problem under loss aversion in an infinite horizon. We need not consider the terminal wealth downside constraint since we only consider an infinite horizon case. However, compared with a finite horizon, optimal consumption and portfolio selection problem in an infinite horizon requires some additional technicalities because the convergence of infinite integral has to be discussed.

In standard consumption and investment models where investors are assumed to behave rationally and maximize expected utility, Zariphopoulou [18] examined a general investment and consumption problem for a single agent who consumes and invests in a riskless asset and a risky one, and with binding trading constraints, limited borrowing, and no bankruptcy, she proved that the value function was the unique smooth solution to the associated Hamilton–Jacobi–Bellman equation and provided the optimal consumption and portfolio policies in feedback form. Lakner and Nygren [9] solved the portfolio optimization problem with both consumption and terminal wealth downside constraints using the gradient operator and the Clark–Ocone formula in Malliavin calculus on a finite horizon. Shin, et al. [13] studied a general optimal consumption and portfolio selection problem of an infinitely-lived investor whose consumption rate process is subjected to downside constraint. Supposing that consumption can never fall below a fixed proportion of the running maximum of past consumption, Arun [1] solved the Merton problem. Therefore, based on these studies, we assume the consumption rate process of a loss averse investor is subjected to a downside constraint. We derive the optimal portfolio and consumption policies in explicit forms. Moreover, we compare the optimal consumption and portfolio policies for loss averse investor with the case of CRRA investor by some numerical results to demonstrate the influence of loss aversion.

The remainder of this paper is organized as follows. In Section 2, we formulate the optimal portfolio and consumption model under loss aversion. Section 3 solves the optimal portfolio and consumption problem with the downside consumption constraints, and gives the closed-form solutions for the portfolio and consumption policies. Section 4 presents some numerical results to compare the optimal consumption and portfolio policies for loss averse investor with the case for CRRA investor. Finally, Section 5 concludes this work.

2. Problem formulation

We consider an infinite horizon \( t \in (0, \infty) \). Let \( (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}) \) be a complete filtered probability space. On this space we define an \( n \)-dimensional standard \( \mathcal{F}_t \)-adapted Brownian motion \( W_t = (W^1_t, \ldots, W^n_t)^T \) with \( W_0 = 0 \), where the filtration \( \mathcal{F}_t = \sigma(W_\tau : 0 \leq \tau \leq t) \). Here and throughout this paper \( A^T \) denotes the transpose of matrix \( A \).

Suppose that there is a frictionless market in which \( n + 1 \) assets are traded continuously. One of the assets \( S_0(t) \) is riskless (for example: bank account). Here we suppose its dynamics is governed by:

\[
dS_0(t) = rS_0(t)\,dt, \quad S_0(0) = S_0.
\]

The remaining assets are risky (for example: stocks), their prices \( S_i(t), \quad i = 1, \ldots, n, \) are modeled by Ito processes:

\[
dS_i(t) = S_i(t) \left[ \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW^j_t \right], \quad S_i(0) = S_i, \quad i = 1, \ldots, n,
\]

where the risk-free rate process \( r \), \( n \)-dimensional mean rate of return process \( \mu(t) = (\mu_1, \ldots, \mu_n)^T \) and \( n \times n \)-matrix valued volatility process \( \sigma = (\sigma_{ij})_{n \times n} \) are constant.

It is well known that a complete market implies the existence and uniqueness of a state price density \( \xi \), given by

\[
\xi_t = \exp \left(- (r + \gamma) t - \kappa^T W_t \right),
\]

where \( \kappa = \sigma^{-1}(\mu - rI) \) denotes the market prices of the risky assets, and \( \gamma = \frac{1}{2} \| \kappa \|^2 \). We can rewrite the state price density process as

\[
\tilde{\xi}_t = -\xi_t (\kappa^T W_t), \quad \tilde{\xi}_0 = 1.
\]

Define \( \tilde{\xi}_t^y = ye^{\gamma t} \tilde{\xi}_t \) for \( y > 0 \), then

\[
d \tilde{\xi}_t^y = \tilde{\xi}_t^y (\rho - r) dt + \kappa^T dW_t, \quad \tilde{\xi}_0 = y.
\]

Suppose that \( X_t \) is the total wealth of the agent at time \( t \). We require \( X_t \geq 0 \) for \( t \in (0, +\infty) \). Let \( \pi_t = (\pi^1_t, \ldots, \pi^n_t)^T \) be the fraction invested on the risky assets and \( C_t \) be the consumption rate at time \( t \). Thus, the fraction of \( 1 - \sum_{i=1}^{n} \pi^i_t \) will be invested on the riskless asset. Then, given initial wealth \( X_0 > 0 \), the wealth process can be written as:

\[
dx_t = X_t (r + (\mu - rI)^T \pi_t - C_t) dt + X_t \pi_t^T \sigma dW_t.
\]
Definition 1. A consumption and portfolio processes pair \((C_t, \pi_t)\) on the infinite planning horizon is admissible at \(x_0\), and write \((C_t, \pi_t) \in A(x_0)\) if it satisfies

1. \(\pi_t\) and \(C_t\) are \(\mathcal{F}_t\)-progressively measurable;
2. the wealth process \(X_t\) corresponding to \(x_0\), \(C_t, \pi_t\) satisfies

\[ X_t \geq 0, \quad 0 \leq t < +\infty, \quad a.s. \]

3. \(\mathbb{E}\int_0^{\infty} \min[0, U(t, C_t^\theta)] dt > -\infty.\)

In this paper, the investor is assumed to treat gains and losses differently, that is, he is risk-averse on gains and risk-seeking on losses. In other words, we suppose \(U(\cdot)\) is an S-shaped utility function:

\[
U(x) = \begin{cases} 
  u_1(x), & \text{if } x > 0, \\
  u_2(x), & \text{if } x \leq 0, 
\end{cases}
\]

where \(u_1(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+\) is strictly increasing, strictly concave and twice differential, with \(u_1(0) = 0\), \(u_1'(0+) = +\infty\), \(u_1''(0 +) = 0\), and \(u_2(\cdot) : \mathbb{R}^- \mapsto \mathbb{R}^-\) is strictly increasing and convex with \(u_2(0) = 0\), \(u_2'(0-) = +\infty\), \(u_2''(-\infty) = 0\). Moreover, we assume \(u_2'(0) > u_1'(0)\) for all \(x > 0\), which reflects that the investor is more sensitive to losses than to gains.

Our goal is then to choose the optimal portfolio and consumption policies \((C_t, \pi_t) \in A(x)\) to maximize the expected discounted S-shaped utility of consumption, in other words, to maximize the objective function

\[
J(\pi_t, C_t) = \mathbb{E}\int_0^{\infty} e^{-\rho t} U(C_t - \theta) dt, \quad (4)
\]

where \(\rho > 0\) is the discount rate, \(U(\cdot)\) is the utility function of consumption, \(\theta\) is the reference level of consumption, which is an inflection point of the utility function \(U(\cdot)\).

In practice, an investor measures his satisfaction by the comparison between his current consumption and a reference level rather than just by his current consumption. The reference level, which is determined by the investor himself, can be influenced by past consumption, aspirations, norms, and social comparisons. For convenience of calculations, in this work we assume that the reference level \(\theta\) is a non-negative constant. In fact, if it is a general \(\mathcal{F}_t\)-measurable random variable (instead of a constant), we can reduce the optimal problem to one with a zero reference level by regarding \(C_t - \theta\) as a new consumption rate. And then repeating the following derivation, we can easily obtain the corresponding conclusion.

Let

\[
V(x_0) = \max_{(G_t, \pi_t) \in A(x_0)} \mathbb{E}\int_0^{\infty} e^{-\rho t} U(C_t - \theta) dt, \quad (5)
\]

be the value function.

3. Optimal consumption and portfolio policies

In this paper, we suppose that there is a downside consumption constraint on the consumption rate process \(C_t\), which means \(C_t \geq \overline{C}\). The optimization problem under loss aversion is reduced to the classical problem with the concave utility function if \(\overline{C} > \theta\). Moreover, in general, a loss averse investor is not content with just the purchase of necessities, the reference level is his consumption aspiration. Thus we assume that \(0 < \overline{C} \leq \theta\). Now the optimization problem can be rewritten as the following form:

\[
\max_{(G_t, \pi_t) \in A(x)} \mathbb{E}\int_0^{\infty} e^{-\rho t} U(C_t - \theta) dt \\
\text{s.t.} \quad dX_t = \left( r + (\mu - r)\pi_t^T - C_t \right) dt + \sigma X_t \pi_T^T dW_t \\
X_0 = x_0, \quad C_t \geq \overline{C}, \quad X_t \geq 0, \quad \forall \ t \geq 0. \quad (6)
\]

Using the martingale method, the dynamic problem (6) can be transformed into the following equivalent static optimization problem:

\[
\max_{(G_t, \pi_t) \in A(x)} \mathbb{E}\int_0^{\infty} e^{-\rho t} U(C_t - \theta) dt \\
\text{s.t.} \quad \mathbb{E}\int_0^{\infty} C_t \xi_t dt = x_0 \\
C_t \geq \overline{C}. \quad (7)
\]

We define \(\xi^*\) as the unique solution to the equation

\[
u_1((u_1')^{-1}(x)) - x \cdot (u_1')^{-1}(x) + (\overline{C} - \theta)x - u_2(\overline{C} - \theta) = 0, \quad (8)
\]
and let

\[ X_\infty(y) = E \int_0^\infty \xi_t[(u'_t)^{-1}(\bar{\xi}_t^Y) + \theta]l_{[\bar{\xi}_t^Y \leq \xi]} dt, \]

\[ Y_\infty(y) = E \int_0^\infty \xi_t[l_{[\bar{\xi}_t^Y > \xi]}] dt, \]

\[ Z_\infty(y) = X_\infty(y) + C Y_\infty(y). \]  

(9)

We can prove the following theorem using the replication technique, which presents the optimal consumption policy and the optimal wealth process of the problem (7):

**Theorem 1.**

1. If \( r x_0 < \bar{C} \), the problem (7) has no solution that satisfies all of the constraints.
2. If \( r x_0 \geq \bar{C} \) and \( Z_\infty(y) < +\infty \), then the optimal policy for the optimization problem (7) is

\[ C_t = \begin{cases} \theta + (u'_t)^{-1}(\bar{\xi}_t^Y), & \text{if } \bar{\xi}_t^Y \leq \bar{\xi}^*; \\ C, & \text{if } \bar{\xi}_t^Y > \bar{\xi}^*. \end{cases} \]  

(10)

and the corresponding optimal process \( X_t \) to \( C_t \) is

\[ X_t = \frac{1}{\xi_t} E \left( \int_t^\infty \xi(s)(u'_s)^{-1}(\bar{\xi}_s^Y)l_{[\bar{\xi}_s^Y \leq \xi]} + C l_{[\bar{\xi}_s^Y > \xi]} \right) ds | F_t \).  

(11)

where \( y \geq 0 \) satisfies

\[ Z_\infty(y) = x_0. \]

**Proof.** 1. Since \( \bar{C} \) is the lower bound of consumption process \( C_t \), we find

\[ E \int_0^\infty \xi_t C_t du \geq \bar{C} E \int_0^\infty \xi_t du \]

\[ = \bar{C} \int_0^{+\infty} \int_0^\infty \exp(-(r + \gamma)t - \kappa T) \cdot \frac{1}{\sqrt{2\pi t}} dT dt \]

\[ = \bar{C} \int_0^{+\infty} \int_0^\infty \exp(-\kappa T) \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}(r + \gamma)t} dt dx. \]

Taking account of the following equality,

\[ \int_0^{+\infty} \frac{1}{\sqrt{\pi t}} e^{-\frac{1}{2}r^2 t} dt = \frac{e^{-\frac{a}{2p}}}{\sqrt{p}}, \quad \forall \, a > 0, \quad \forall \, p > 0, \]

thus if \( \bar{C} > r x_0 \), we can obtain

\[ E \int_0^\infty \xi_t C_t du > \bar{C} \int_0^{+\infty} \exp(-\kappa T) \cdot \frac{1}{\sqrt{2(r + \gamma)}} e^{-\sqrt{2(r + \gamma)}|\xi|} dx \]

\[ = \bar{C} x_0. \]

Apparently, this is contradictory with Theorem 9.4 of Karatzas and Shreve [7] (p. 137), i.e., \( E \int_0^\infty \xi_t C_t du = x_0 \). Economically, if \( \bar{C} > r x_0 \), the interest flow is insufficient to maintain consumption (or the perpetuity value of maintaining consumption at its previous level is larger than initial wealth). So the problem (7) is infeasible.

2. Let

\[ \bar{U}(C_t) = e^{-\rho t} U(C_t - \theta) - y_C \xi_t, \]  

(12)

we can easily see that if \( C_t \) satisfies the constraint conditions in (7) and makes \( \bar{U} \) reaching the maximum value, then \( C_t \) is the optimal solution of problem (7).

In fact, for all \( C_t \) satisfying the constraint conditions in (7), we have

\[ E \int_0^\infty e^{-\rho t} U(C_t - \theta) dt - E \int_0^\infty e^{-\rho t} U(C_t - \theta) dt \]

\[ = E \int_0^\infty \left[ e^{-\rho t} U(C_t - \theta) - y_C \xi_t \right] dt \geq 0. \]

Now, similar to the method of Berkelaar et al. [3], we solve the maximization problem (12) segmentally, since the utility function \( U(\cdot) \) is not concave in the whole interval. First we need to get the local optimal solution.
If $C_t \leq \theta$, $U(C_t - \theta)$ is convex, and (12) reaches the maximum value when $C_{t+1} = \mathcal{C}$ or $C_{t+2} = \theta$. On the other hand, if $C_t > \theta$, $U(C_t - \theta)$ is concave, and there exists a unique optimal solution $C_{t,3}$ satisfying the following conditions:

$$
\begin{align*}
& e^{-\rho t} u'_1(C_{t,3} - \theta) - y\xi_t + \lambda = 0, \\
& \lambda C_{t,3} = 0.
\end{align*}
$$

Thus

$$
C_{t,3} = (u'_1)^{-1}(y e^{\rho t} \xi_t + \theta).
$$

Next, in order to determine the global optimal solution, all we need to do is to compare the local optimal solutions. Firstly, because $u_1(x)$ is concave, we have

$$
\begin{align*}
\hat{U}(C_{t,3}) - \hat{U}(\mathcal{C}) = e^{-\rho t} u_1((u'_1)^{-1}(y e^{\rho t} \xi_t)) - y\xi_t \cdot ((u'_1)^{-1}(y e^{\rho t} \xi_t) + \theta) + y\theta \\
= e^{-\rho t} u_1((u'_1)^{-1}(y e^{\rho t} \xi_t)) - e^{-\rho t} u_1(0) - y\xi_t \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) \\
= e^{-\rho t} u'_1(\tau) \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) - y\xi_t \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) \\
\geq e^{-\rho t} u'_1((u'_1)^{-1}(y e^{\rho t} \xi_t)) \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) - y\xi_t \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) \\
= 0,
\end{align*}
$$

for some $\tau \in (0, (u'_1)^{-1}(y e^{\rho t} \xi_t))$. Hence, $C_{t,2} = \theta$ is never the optimal solution.

And secondly,

$$
\begin{align*}
\hat{U}(C_{t,3}) - \hat{U}(\mathcal{C}) = & e^{-\rho t} u_1((u'_1)^{-1}(y e^{\rho t} \xi_t)) - y\xi_t \cdot ((u'_1)^{-1}(y e^{\rho t} \xi_t) + \theta) - e^{-\rho t} u_2(\mathcal{C} - \theta) + y\mathcal{C} \\
= & e^{-\rho t} u_1((u'_1)^{-1}(y e^{\rho t} \xi_t)) - e^{-\rho t} u_1(0) - y\xi_t \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) + \mathcal{C} - e^{-\rho t} u_2(\mathcal{C} - \theta) \\
\geq & e^{-\rho t} u'_1((u'_1)^{-1}(y e^{\rho t} \xi_t)) \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) - y\xi_t \cdot (u'_1)^{-1}(y e^{\rho t} \xi_t) + \mathcal{C} - e^{-\rho t} u_2(\mathcal{C} - \theta) \\
= & -e^{-\rho t} (\theta - \mathcal{C}) y e^{\rho t} \xi_t + u_2(\mathcal{C} - \theta).
\end{align*}
$$

Let

$$
g(y e^{\rho t} \xi_t) = e^{\rho t} [\hat{U}(C_{t,3}) - \hat{U}(\mathcal{C})],$$

that is,

$$
g(x) = u_1((u'_1)^{-1}(x)) - x[(u'_1)^{-1}(x) + \theta - \mathcal{C}] - u_2(\mathcal{C} - \theta).
$$

Evidently though, if $g(y e^{\rho t} \xi_t) > 0$, then $\hat{U}(C_{t,3}) > \hat{U}(\mathcal{C})$, that is, $C_{t,3}$ is the optimal solution. Otherwise, $\mathcal{C}$ is the optimal solution.

On the one hand, it can be easily proved that

$$
g(x) = \mathcal{C} - \theta - (u'_1)^{-1}(x) \leq 0,
$$

that is, $g(x)$ is strictly decreasing, and $g(x) \to -\infty$ when $x \to +\infty$. On the other hand, (13) implies that

$$
g\left(\frac{u_2(\mathcal{C} - \theta)}{\mathcal{C} - \theta}\right) \geq -(\theta - \mathcal{C}) \cdot \frac{u_2(\mathcal{C} - \theta)}{\mathcal{C} - \theta} - u_2(\mathcal{C} - \theta) = 0.
$$

Therefore, $g(x)$ has a unique null point in $(\frac{u_2(\mathcal{C} - \theta)}{\mathcal{C} - \theta}, +\infty)$. We denote this null point by $\xi^*$. To sum up, when $y e^{\rho t} \xi_t < \xi^*$, $C_{t,3}$ is the optimal solution, otherwise, $\mathcal{C}$ is the optimal solution. That is to say, the optimal consumption policy for the loss averse investor is

$$
C_t' = \begin{cases} 
\theta + (u'_1)^{-1}(y e^{\rho t} \xi_t), & \text{if } y e^{\rho t} \xi_t \leq \xi^*, \\
\mathcal{C}, & \text{if } y e^{\rho t} \xi_t > \xi^*.
\end{cases}
$$

According to the theorem 9.11 of Karatzas and Shreve [7] (p. 141), if $Z_\infty(y) < +\infty$, then the optimal wealth process is

$$
X_t = \frac{1}{\xi_t} E\left(\int_t^{\infty} \xi_s C_s ds \mid F_t\right) = \frac{1}{\xi_t} E\left(\int_t^{\infty} \xi_s \left[ (u'_1)^{-1}(\xi^*_s) \cdot I_{\{\xi_s \leq \xi^*_s\}} + \mathcal{C}_{I_{\{\xi_s > \xi^*_s\}}} \right] ds \mid F_t\right).
$$

Now we give an equivalent condition for $X_\infty(y) < +\infty$, and the expressions of $X_\infty(y)$ and $Y_\infty(y)$.

**Lemma 2.** The condition

$$
\int_0^1 \eta^{-\xi} \nu(\eta) d\eta < +\infty
$$
is equivalent to $\mathcal{X}_\infty(y) < +\infty$, and under this condition, $\forall y > 0,$

$$
\mathcal{X}_\infty(y) = \begin{cases} 
\frac{y^{\frac{1}{1-\delta_2}}}{\gamma(\delta_2 - 1)} \int_y^{\xi^*} \eta^{-\delta_2} v(\eta) d\eta + \frac{y^{\frac{1}{1-\delta_2}}}{\gamma(\delta_2 - 1)} \int_y^{\xi^*} \eta^{-\delta_1} v(\eta) d\eta, & \text{if } 0 < y \leq \xi^*; \\
\frac{y^{\frac{1}{1-\delta_2}}}{\gamma(\delta_2 - 1)} \int_0^{\xi^*} \eta^{-\delta_2} v(\eta) d\eta, & \text{if } y > \xi^*.
\end{cases}
$$

$$
\mathcal{Y}_\infty(y) = \begin{cases} 
\frac{1}{\gamma(1-\delta_2)} \left[ \frac{1}{\delta_1 - 1} \frac{1}{\delta_2 - 1} \right] \left( \frac{\gamma}{\xi^*} \right)^{\delta_1 - 1}, & \text{if } 0 < y \leq \xi^*; \\
\frac{1}{\gamma(1-\delta_2)} \left[ \frac{1}{\delta_1 - 1} + \frac{1}{\delta_2 - 1} \right] \left( \frac{\gamma}{\xi^*} \right)^{\delta_1 - 1}, & \text{if } y > \xi^*.
\end{cases}
$$

where

$$
v(x) = (u'_i)^{-1}(x) + \theta, \quad \tilde{r} = r + \gamma - \rho.
$$

$$
\delta_i = \frac{\tilde{r}}{2\gamma} - (-1)^i \frac{\sqrt{4\rho \gamma + \tilde{r}^2}}{2\gamma}, \quad i = 1, 2.
$$

**Proof.** We only compute $\mathcal{X}_\infty(y)$. Similarly, we can obtain the expression of $\mathcal{Y}_\infty(y)$.

We can see that

$$
\mathcal{X}_\infty(y) = E \int_0^{\xi^*} \frac{1}{\gamma(1-\delta_2)} \left[ (u'_i)^{-1}(ye^{\delta_1 - 1} t) + \theta \right] \cdot l_{(ye^{\delta_1 - 1} \leq \xi^* - 1)} dt
$$

$$
= \int_{-\infty}^{\xi^*} \int_0^{\xi^*} \exp(-(r + \gamma)t - k^T x) - v(\gamma \exp(-\tilde{r} t - k^T x))
\cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} \cdot l_{[-\tilde{r} t - k^T x \leq \ln \xi^* - \ln(y)]} dt dx.
$$

Define

$$
z = \tilde{r} t + k^T x,
$$

then

$$
\mathcal{X}_\infty(y) = \int_{-\infty}^{\xi^*} \int_0^{\xi^*} e^{-\frac{z^2}{2t}} \cdot v(\gamma z) l_{[\gamma z \leq \ln \xi^* - \ln(y)]} \cdot \frac{1}{\sqrt{4\pi \gamma t}} e^{-\frac{(\tilde{r}^2 + \rho)^t}{4\pi \gamma t}} dt dz.
$$

Taking account of the following equality,

$$
\int_0^{+\infty} \frac{1}{\sqrt{\pi t}} e^{-\frac{z^2}{t}} dt = \frac{e^{-\sqrt{\pi p}}}{\sqrt{\pi p}}, \quad \forall \ a > 0, \ \forall \ p > 0,
$$

we can get

$$
\mathcal{X}_\infty(y) = \frac{1}{\sqrt{4\rho \gamma + \tilde{r}^2}} \int_{-\infty}^{\xi^*} \exp \left( \frac{\tilde{r}^2 - 2\gamma z - \sqrt{4\rho \gamma + \tilde{r}^2}}{2\gamma} \right) \cdot v(\gamma z) l_{[\gamma z \leq \ln \xi^* - \ln(y)]} dz.
$$

From the definitions of $\delta_1$ and $\delta_2$, we can get $\delta_1 > 0$ and $\delta_2 < 0$. Therefore,

$$
\mathcal{X}_\infty(y) = \frac{1}{\gamma(1-\delta_2)} \left[ \int_0^{+\infty} e^{(\delta_1 - 1)z} \cdot v(\gamma z) \cdot l_{[\gamma z \leq \ln \xi^* - \ln(y)]} dz 
\right]
$$

$$
+ \left[ \int_0^{+\infty} e^{(\delta_1 - 1)z} \cdot v(\gamma z) \cdot l_{[\gamma z \leq \ln \xi^* - \ln(y)]} dz \right]
$$

$$
= \frac{1}{\gamma(1-\delta_2)} \left[ \int_0^{+\infty} y^{\delta_1 - 1} \eta^{-\delta_1} v(\eta) l_{[\eta \leq \xi^*]} d\eta + \int_y^{+\infty} y^{\delta_1 - 1} \eta^{-\delta_1} v(\eta) l_{[\eta \geq \xi^*]} d\eta \right]
$$

$$
= \frac{1}{\gamma(1-\delta_2)} \left[ \int_0^{\xi^*} y^{\delta_2 - 1} \eta^{-\delta_2} v(\eta) d\eta l_{[y \geq \xi^*]} \right]
$$

$$
+ \left( y^{\delta_1 - 1} \int_0^{\xi^*} \eta^{-\delta_1} v(\eta) d\eta + y^{\delta_1 - 1} \int_y^{+\infty} \eta^{-\delta_1} v(\eta) d\eta \right) l_{[y \geq \xi^*]}\right].
$$

Since

$$
\int_0^{1} \eta^{-\delta_2} v(\eta) d\eta < +\infty
$$
is equivalent to
\[
\int_0^\epsilon \eta^{-\delta_2} v(\eta) d\eta < +\infty, \ \forall \ \epsilon > 0,
\]
we have \( \lambda'_\infty(y) < +\infty \) if and only if
\[
\int_0^1 \eta^{-\delta_2} v(\eta) d\eta < +\infty.
\]

Thus, \( Z'_\infty(y) < +\infty \) is equivalent to \( \lambda'_\infty(y) < +\infty \). What’s more, we can derive, by Feynman–Kac formula,
\[
\gamma^2 Z'_\infty(y) + (3 \gamma - \tilde{\gamma}) y Z'_\infty(y) - r Z'_\infty(y) + v(y) I_{y \in [\xi, \infty)} + \overline{C} I_{y > \xi} = 0.
\]

**Proposition 3.** \( Z'_\infty(y) \) is strictly decreasing, and
\[
\lim_{y \to 0^+} Z'_\infty(y) = +\infty, \quad \lim_{y \to +\infty} Z'_\infty(y) = 0.
\]

**Proof.** From Eq. (8), we find
\[
(u'_1)^{-1}(\xi^*) > \overline{C} - \theta.
\]
We can see that \((u'_1)^{-1}(\cdot) : R^+ \mapsto R^+ \) is strictly decreasing and continuous because \( u_1(\cdot) : R^+ \mapsto R^+ \) is strictly increasing, concave and twice differentiable. Thus
\[
(u'_1)^{-1}(\eta) > \overline{C} - \theta, \quad \forall \ 0 < \eta < \xi^*.
\]

From **Lemma 2**, we can obtain
\[
\lambda'_\infty(y) = \left\{ \begin{array}{ll}
\frac{(\delta_2 - 1) y^{\delta_2 - 2}}{y(\delta_1 - \delta_2)} \int_0^\xi \eta^{-\delta_2} v(\eta) d\eta + \frac{(\delta_2 - 1) y^{\delta_2 - 2}}{y(\delta_1 - \delta_2)} \int_0^{\xi^*} \eta^{-\delta_1} v(\eta) d\eta, & \text{if } 0 < y \leq \xi^*; \\
\frac{(\delta_2 - 1) y^{\delta_2 - 2}}{y(\delta_1 - \delta_2)} \int_0^{\xi^*} \eta^{-\delta_1} v(\eta) d\eta, & \text{if } y > \xi^*.
\end{array} \right.
\]
\[
\gamma'_\infty(y) = \left\{ \begin{array}{ll}
\frac{1}{y(\delta_1 - \delta_2)^2} \left( \frac{y}{\xi} \right)^{\delta_1 - 2}, & \text{if } 0 < y \leq \xi^*; \\
\frac{1}{y(\delta_1 - \delta_2)^2} \left( \frac{y}{\xi} \right)^{\delta_2 - 2}, & \text{if } y > \xi^*.
\end{array} \right.
\]
Then when \( y > \xi^* \),
\[
Z'_\infty(y) = \lambda'_\infty(y) + \overline{C} \gamma'_\infty(y)
\]
\[
= \frac{(\delta_2 - 1) y^{\delta_2 - 2}}{y(\delta_1 - \delta_2)} \int_0^{\xi^*} \eta^{-\delta_2} v(\eta) d\eta + \frac{\overline{C}}{y(\delta_1 - \delta_2)^2} \left( \frac{y}{\xi^*} \right)^{\delta_1 - 2}
\]
\[
= \frac{y^{\delta_2 - 2}}{y(\delta_1 - \delta_2)} \left( (\delta_2 - 1) \int_0^{\xi^*} \eta^{-\delta_1} v(\eta) d\eta + \overline{C} \xi^{*(1-\delta_1)} \right)
\]
\[
= \frac{y^{\delta_2 - 2}}{y(\delta_1 - \delta_2)} \left[ \overline{C} - \theta - (u'_1)^{-1}(\eta) \right] d\eta^{1-\delta_2}
\]< 0.
\]
Similarly, when \( 0 < y < \xi^* \),
\[
(\delta_1 - 1) \int_0^{\xi^*} \eta^{-\delta_1} v(\eta) d\eta + \overline{C} \xi^{*(1-\delta_1)} < 0.
\]
And since \( \delta_2 > 0 \), we can easily find
\[
Z'_\infty(y) < 0.
\]
So, \( Z'_\infty(y) \) is strictly decreasing. In addition, from **Lemma 2**, we can observe
\[
\lim_{y \to 0^+} \lambda'_\infty(y) = +\infty, \quad \lim_{y \to +\infty} \lambda'_\infty(y) = 0.
\]
\[
\lim_{y \to 0^+} \gamma'_\infty(y) = 0, \quad \lim_{y \to +\infty} \gamma'_\infty(y) = 0.
\]
Thus
\[
\lim_{y \to 0^+} Z'_\infty(y) = +\infty, \quad \lim_{y \to +\infty} Z'_\infty(y) = 0.
\]
Define
\[
V_1(y) = E \int_0^{y} e^{-\rho t} u_1((u_1^\prime)^{-1}(\tilde{\xi}_t^y))I_{\{\tilde{\xi}_t^y \leq \xi^*\}} \, ds,
\]
\[
V_2(y) = E \int_0^{\infty} e^{-\rho t} I_{\{\tilde{\xi}_t^y \leq \xi^*\}} \, ds.
\]

Similar to Lemma 2, we can obtain

**Lemma 4.**
\[
V_1(y) = \begin{cases} \frac{y^\gamma}{\gamma(\delta_1-\delta_2)} \int_0^y \eta^{-\delta_2-1} u_1((u_1^\prime)^{-1}(\eta)) \, d\eta + \frac{y^\gamma}{\gamma(\delta_1-\delta_2)} \int_y^{\infty} \eta^{-\delta_2-1} u_1((u_1^\prime)^{-1}(\eta)) \, d\eta, & \text{if } 0 < y \leq \xi^*; \\ \frac{y^\gamma}{\gamma(\delta_1-\delta_2)} \int_0^{\xi^*} \eta^{-\delta_2-1} u_1((u_1^\prime)^{-1}(\eta)) \, d\eta, & \text{if } y > \xi^*; \end{cases}
\]
\[
V_2(y) = \begin{cases} \frac{1}{\gamma(\delta_1-\delta_2)} \left[ \frac{1}{\delta_1} - \frac{1}{\delta_2} + \frac{1}{\delta_2} \left( \frac{\gamma}{\xi^2} \right)^{\delta_2} \right], & \text{if } 0 < y \leq \xi^*; \\ \frac{1}{\gamma(\delta_1-\delta_2)} \left[ \frac{1}{\delta_1} - \frac{1}{\delta_2} + \frac{1}{\delta_2} \left( \frac{\gamma}{\xi^2} \right)^{\delta_2} \right], & \text{if } y > \xi^*. \end{cases}
\]

Then the value function:
\[
V(x_0) = E \int_0^{\infty} e^{-\rho t} \tilde{C}_t \, ds = V_1(y) + u_2(\bar{C} - \theta) \cdot V_2(y),
\]
where \( y \geq 0 \) satisfies \( \bar{Z}_\infty(y) = x_0 \).

According to Theorem 1 and Corollary 9.15 of Karatzas and Shreve [7] (p. 144), we can derive the optimal consumption and portfolio policies and the optimal wealth process of the problem (7):

**Theorem 5.** If \( rx_0 \geq \bar{C} \) and \( \bar{Z}_\infty(y) < +\infty \), then the optimal wealth process for problem (7) is given by
\[ X_t = \bar{Z}_\infty(\tilde{\xi}_t^x), \]
and the feedback form solutions for the optimal portfolio and consumption policies are respectively
\[
\begin{align*}
C_t &= \left[ \theta + (u_1^\prime)^{-1}(\tilde{\xi}_t^x) \right] I_{\{\tilde{\xi}_t^x \leq \xi^*\}} + \bar{C}_t I_{\{\tilde{\xi}_t^x > \xi^*\}}; \\
\pi_t &= -\frac{\sigma^{-1} \tilde{\xi}_t^x X_\infty(\tilde{\xi}_t^x)}{\bar{Z}_\infty(\tilde{\xi}_t^x)}.
\end{align*}
\]
The value function of the loss averse investor with a downside consumption constraint is
\[ V(x_0) = V_1(y) + u_2(\bar{C} - \theta) \cdot V_2(y), \]
where \( y \geq 0 \) satisfies \( \bar{Z}_\infty(y) = x_0 \).

Now we take \( \bar{C} = 0 \), which means \( C_t \geq 0 \). Namely, the optimization problem with a consumption constraint becomes the problem without consumption constraint that is exactly what we have discussed in [14]. Thus we obtain the following corollary.

**Corollary 6.** Suppose that \( \xi^* \) is the unique solution to the equation
\[ u_1((u_1^\prime)^{-1}(\xi^*)) = \lambda((u_1^\prime)^{-1}(\xi^*) + \theta) - u_2(-\theta) = 0. \]
If \( \bar{X}_\infty(y) < +\infty \), the feedback form solutions for the optimal portfolio and consumption policies for the loss averse investor without consumption constraints are respectively
\[
\begin{align*}
C_t &= \left[ \theta + (u_1^\prime)^{-1}(\tilde{\xi}_t^x) \right] I_{\{\tilde{\xi}_t^x \leq \xi^*\}}; \\
\pi_t &= -\frac{\sigma^{-1} \tilde{\xi}_t^x X_\infty(\tilde{\xi}_t^x)}{\bar{X}_\infty(\tilde{\xi}_t^x)}.
\end{align*}
\]
the optimal wealth process
\[ X(t) = \frac{1}{\xi^*} E \left( \int_t^{\infty} \tilde{\xi}_s [ (u_1^\prime)^{-1}(\tilde{\xi}_t^x) + \theta ] I_{\{\tilde{\xi}_s \leq \xi^*\}} \, ds | F_t \right) = X_\infty(\tilde{\xi}_t^x), \]
and the value function
\[ V(x_0) = V_1(y) + u_2(-\theta) \cdot V_2(y), \]
where \( y \geq 0 \) satisfies
\[ X_\infty(y) = x_0. \]
4. Comparison with the case of risk aversion

In this section, in order to demonstrate the effectiveness of our results, we compare the optimal consumption and portfolio policies for loss averse investor with the case of classical risk averse investor by some numerical results. We take the two-piece power utility function proposed by Tversky and Kahneman [17] as an example. Its form is as follows:

\[
U(x) = \begin{cases} 
  u_1(x) = x^\alpha, & \text{if } x > 0; \\
  u_2(x) = -\lambda (-x)^\beta, & \text{if } x \leq 0,
\end{cases}
\]

where \(0 < \alpha \leq \beta < 1, \lambda > 1, \lambda\) describes the coefficient of loss aversion. In the classical risk aversion model, the corresponding utility function is the constant relative risk aversion utility function \(u(x) = x^\alpha\). We can see that \(U(x)\) is S-shaped and increasing, while \(u(x)\) is concave and increasing, see Fig. 1.

Define

\[
\delta_3 = \frac{1}{\alpha - 1} - \delta_1 + 1, \quad \delta_4 = \frac{1}{\alpha - 1} - \delta_2 + 1.
\]

it is obvious that \(\delta_3 < 0\), and

\[
\int_0^1 \eta^{-\delta_1} (u'_1)^{-1}(\eta) + \theta \, d\eta = \frac{\alpha - 1}{\delta_4} + \frac{\theta}{1 - \delta_2} < +\infty
\]

for \(\delta_4 > 0\). Thus, now let us suppose \(\delta_4 > 0\). Based on Theorem 5 and its corollary, we can obtain the following conclusions:

(1)

\[
\gamma'(\delta_1 - \delta_2) \lambda_1(\bar{y}) = \begin{cases} 
  a_1\tilde{y}^{\delta_1-1} + a_3\tilde{y}^{\frac{1}{\delta_1}} + a_4\bar{y}, & \text{if } 0 < \tilde{y} \leq \xi^*; \\
  a_2\tilde{y}^{\delta_2-1}, & \text{if } \tilde{y} > \xi^*. \end{cases}
\]

(2) The optimal consumption and portfolio policies for the case where the loss averse investor does not have a downside consumption constraint are respectively

\[
C^*_t = \begin{cases} 
  \theta + \left(\frac{\xi^*_t}{\alpha}\right)^{\frac{1}{\delta}}, & \text{if } 0 < \xi^*_t \leq \xi^*; \\
  0, & \text{if } \xi^*_t > \xi^*;
\end{cases}
\]

(20)
and

\[
\pi_t^* = \begin{cases} 
\sigma^{-1} \kappa \frac{\mu_l}{\alpha_3 \xi^1 + \alpha_0 (1 - \delta_1)}, & \text{if } 0 < \xi_t \leq \xi^*; \\
\sigma^{-1} \kappa (1 - \delta_2), & \text{if } \xi_t > \xi^*. 
\end{cases}
\]  

(21)

The value function of the loss averse investor without a downside consumption constraint is

\[
\gamma(\delta_1 - \delta_2) V(y) = \begin{cases} 
\left( \frac{\xi^1}{\delta_1} + \frac{\alpha_1}{\alpha} \right) y^{\delta_1} + \frac{\alpha_1}{\alpha} y^{\delta_1}, & \text{if } 0 < y \leq \xi^*; \\
\left( \frac{\xi^1}{\delta_4} + \frac{\alpha_2}{\alpha} \right) y^{\delta_4} + \left( \frac{1}{\delta_4} - \frac{1}{\delta_2} \right) y^2 (-\theta), & \text{if } \gamma > \xi^*. 
\end{cases}
\]  

(22)

(3) The optimal consumption and portfolio policies for the loss averse investor with a downside consumption constraint are respectively

\[
\hat{C}_t^* = \begin{cases} 
\theta + (\xi_t) \frac{\alpha}{\alpha^2}, & \text{if } 0 < \xi_t \leq \xi^*; \\
\xi^*, & \text{if } \xi_t > \xi^*. 
\end{cases}
\]  

(23)

and

\[
\tilde{\pi}_t^* = \begin{cases} 
\sigma^{-1} \kappa \frac{\mu_l}{\alpha_3 \xi^1 + \alpha_0 (1 - \delta_1)}, & \text{if } 0 < \xi_t \leq \xi^*; \\
\sigma^{-1} \kappa (1 - \delta_2), & \text{if } \xi_t > \xi^*. 
\end{cases}
\]  

(24)

The value function for the loss averse investor with the downside consumption constraints is

\[
\gamma(\delta_1 - \delta_2) V(y) = \begin{cases} 
\left( \frac{\xi^1}{\delta_1} + \frac{\alpha_1}{\alpha} \right) y^{\delta_1} + \frac{\alpha_1}{\alpha} y^{\delta_1}, & \text{if } 0 < \gamma \leq \xi^*; \\
\left( \frac{\xi^1}{\delta_4} + \frac{\alpha_2}{\alpha} \right) y^{\delta_4} + \left( \frac{1}{\delta_4} - \frac{1}{\delta_2} \right) y^2 (\theta - C), & \text{if } \gamma > \xi^*. 
\end{cases}
\]  

(25)

Where \( \tilde{\xi}_t = ye^{\alpha \xi_t}, y \geq 0 \) satisfies \( \lambda_x(y) = \gamma; \hat{\xi}_t = \gamma e^{\alpha \xi_t}, \gamma \geq 0 \) satisfies \( \gamma \lambda_x(\gamma) = \gamma; \hat{\xi}^* \) and \( \hat{\xi}^* \) are the solutions to the following equations respectively

\[
(1 - \alpha) \left( \frac{\alpha}{\alpha} \right) \frac{x^{\delta_1}}{\delta_1} + \theta x + \lambda \theta = 0, \\
(1 - \alpha) \left( \frac{\alpha}{\alpha} \right) \frac{x^{\delta_4}}{\delta_4} + (C - \theta) x + \lambda (\theta - C) = 0;
\]

and

\[
a_1 = \frac{\theta}{1 - \delta_1} e^{(1 - \delta_1)} + \frac{\alpha_1}{\alpha} \frac{\xi^1}{\delta_1}, \quad a_2 = \frac{\theta}{1 - \delta_2} e^{(1 - \delta_2)} + \frac{\alpha_1}{\alpha} \frac{\xi^1}{\delta_2}, \\
a_3 = \frac{1}{\delta_4} - \frac{1}{\delta_2} \frac{\alpha_2}{\alpha}, \quad a_4 = \frac{1}{\delta_4} - \frac{1}{\delta_2} - \frac{1}{\delta_1}, \\
b_1 = \frac{\theta}{1 - \delta_1} e^{(1 - \delta_1)} + \frac{\alpha_1}{\alpha} \frac{\xi^1}{\delta_1}, \quad b_2 = \frac{\theta}{1 - \delta_2} e^{(1 - \delta_2)} + \frac{\alpha_1}{\alpha} \frac{\xi^1}{\delta_2}.
\]  

(26)

Based on [13], we can compute the optimal consumption and portfolio policies for a CRRA investor without or with a consumption constraint. They are the following results.

1. The optimal consumption and portfolio policies for the case where the CRRA investor \( (u(x) = x^\alpha) \) does not have a downside consumption constraint are respectively

\[
\left\{ \begin{array}{ll}
C_t^* = (\hat{\xi}_t^c / \alpha)^{\frac{1}{\alpha-1}}, \\
\pi_t^* = \sigma^{-1} \kappa / (1 - \alpha).
\end{array} \right.
\]  

(27)

The optimal wealth process is given by \( X_{t,c} = \lambda_{x,c}(\hat{\xi}_t^c) \), with

\[
\gamma(\delta_1 - \delta_2) X_{t,c}(y) = a_3 \frac{y^{\frac{1}{\alpha}}}{\alpha}. 
\]  

(28)

2. The optimal consumption and portfolio policies for the CRRA investor \( (u(x) = x^\alpha) \) with a downside consumption constraint are respectively

\[
\hat{C}_t^* = \begin{cases} 
(\hat{\xi}_t^c / \alpha)^{\frac{1}{\alpha-1}}, & \text{if } 0 < \hat{\xi}_t^c \leq \alpha \gamma^{-1}; \\
\gamma, & \text{if } \hat{\xi}_t^c > \alpha \gamma^{-1}; 
\end{cases}
\]  

(29)
and

\[
\hat{X}_{t, c} = \begin{cases} 
\sigma^{-1} \frac{\frac{\alpha y_t^{\delta_1 - 1} + a_3 \tilde{y}_c^{\frac{1}{\alpha}}}{\delta_1^{\frac{1}{\alpha}} + b_1}}{\frac{\alpha y_t^{\delta_1 - 1} + a_3 \tilde{y}_c^{\frac{1}{\alpha}}}{\delta_1^{\frac{1}{\alpha}} + b_1}}, & \text{if } \hat{\xi}_{t, c} \leq \alpha C^{\alpha - 1}; \\
\sigma^{-1} \frac{\frac{\alpha y_t^{\delta_1 - 1} + a_3 \tilde{y}_c^{\frac{1}{\alpha}}}{\delta_1^{\frac{1}{\alpha}} + b_1}}{\frac{\alpha y_t^{\delta_1 - 1} + a_3 \tilde{y}_c^{\frac{1}{\alpha}}}{\delta_1^{\frac{1}{\alpha}} + b_1}}, & \text{if } \hat{\xi}_{t, c} > \alpha C^{\alpha - 1}.
\end{cases}
\]

(30)

The optimal wealth process is given by \( \hat{X}_{t, c} = Z_{\infty, c}(\hat{\xi}_{t, c}) \), with

\[
y(\delta_1 - \delta_2)Z_{\infty, c}(y) = \begin{cases} 
b_3 \hat{y}_c^{\delta_1 - 1} + a_3 \hat{y}_c^{\frac{1}{\alpha}}, & \text{if } 0 < \hat{y}_c \leq \alpha C^{\alpha - 1}; \\
b_4 \hat{y}_c^{\delta_1 - 1} + a_4 \tilde{C}, & \text{if } \hat{y}_c > \alpha C^{\alpha - 1}.
\end{cases}
\]

(31)

Where \( \hat{\xi}_{t, c} = y_c e^{\alpha t} \hat{\xi}_t \), \( y_c \geq 0 \) satisfies \( X_{\infty, c}(y) = x_0 \); \( \hat{\xi}_{t, c} = \hat{y}_c e^{\alpha t} \xi_t \), \( \hat{y}_c \geq 0 \) satisfies \( Z_{\infty, c}(y) = x_0 \); and

\[
X_{\infty, c}(y) = \mathbb{E} \int_0^\infty \xi_t C_{\xi_t} dt, \quad Z_{\infty, c}(y) = \mathbb{E} \int_0^\infty \hat{\xi}_{t, c} \tilde{C}_{\hat{\xi}_{t, c}} dt,
\]

\[
b_3 = \left( \frac{1}{\delta_1 - 1} + \frac{1}{\delta_3} \right) \alpha^{1 - b_1} \tilde{C}^{\delta_3(\alpha - 1)}, \quad b_4 = \left( \frac{1}{\delta_2 - 1} + \frac{1}{\delta_4} \right) \alpha^{1 - b_2} \tilde{C}^{\delta_4(\alpha - 1)}.
\]

Now, we compare the optimal consumption and portfolio policies for loss averse investor with the case of CRRA investor. We obtain some numerical results. See Figs. 2–6. In these figures, the solid/dashed lines represent the cases for the loss averse investor without/with a downside consumption constraint, respectively, and the dash-dot/dotted lines plot the cases for the CRRA investor without/with a downside consumption constraint, respectively. The basic economic parameter values are taken to be

\[
\alpha = \beta = 0.68, \quad \lambda = 2.25, \quad \theta = 1, \quad \tilde{C} = 0.5,
\]

\[
r = 0.01, \quad \mu = 0.05, \quad \sigma = 0.2, \quad \rho = 0.07.
\]

(32)

According to Fig. 2, the optimal wealth for both loss averse investor and CRRA investor is monotonic decreasing as a function of \( ye^{\alpha t} \hat{\xi}_t \). The optimal wealth for a CRRA investor drops faster than that for a loss averse investor.

Fig. 3 reveals that a loss averse investor likes consuming more wealth than a CRRA investor does since a loss averse investor cares more about how much the consumption exceeds his reference level than the value of the consumption. And he has two different kinds of consumption behavior in the good states (\( ye^{\alpha t} \hat{\xi}_t \leq \hat{\xi}_t^* \)) and the bad states (\( ye^{\alpha t} \hat{\xi}_t > \hat{\xi}_t^* \)) of market.
In the good states of market, both of loss averse investors with/without a downside consumption constraint consume more than the reference level. In the bad states of market, the loss averse investor without a downside consumption constraint does not consume at all, while the loss averse investor with a downside consumption constraint only maintains a minimum consumption \((\bar{C}_t = \bar{C})\). What is more, in the same situation, the size of the good-states region for the loss averse investor with a downside consumption constraint is more than that without a downside consumption constraint.

Combining (27), (21), (29) and (30) with Figs. 2 and 3, we can conclude that the size of the good-states region not only relates to the initial wealth \(x\), the reference level \(\theta\), but also depends on the downside consumption constraint \(\bar{C}\). If \(\theta\) increases, \(x\) decreases, or \(\bar{C}\) decreases, the good-states region decreases. In other words, if the prospective consumption level level of the investor is too high or the initial wealth is not enough, the undersired situation is more likely to happen.

Based on Fig. 4, we can see that the fraction invested in stocks for a loss averse investor is equal to that for a CRRA investor at some critical values, respectively. Comparing with a CRRA investor, a loss averse investor has two different kinds below and above these critical values. In the bad states of market \((y e^{\rho t} \xi_t > \xi^*)\), the loss averse investor without a downside consumption constraint invests a constant fraction in stocks, which is the same as the classical CRRA model. While in the good states of market \((y e^{\rho t} \xi_t \leq \xi^*)\), the optimal fraction invested in stocks is \(V\)-shaped. It is minimal at some critical value. When \(y e^{\rho t} \xi_t\) falls below this critical value, the investor decreases the exposure to stocks. When \(y e^{\rho t} \xi_t\) remains above this critical value, he increases the fraction. When a loss averse investor does have a downside consumption constraint, the fraction invested in stocks is decreasing with respect to \(y e^{\rho t} \xi_t\). What is more, the higher the value of \(y e^{\rho t} \xi_t\), the slower the optimal portfolio policy’s descent. In addition, whether the loss averse investor has a consumption constraint or not, when \(\xi_t\) or \(\bar{\xi}_t\) goes to the two different extreme cases, the optimal portfolio policies go to the optimal portfolio policies of the different CRRA investors respectively. Note that as \(\xi_t\) or \(\bar{\xi}_t\) goes to +\(\infty\), the fraction invested in stocks \(\pi^*_t\) or \(\bar{\pi}_t\) goes to \(\sigma^{-1} \kappa (1 - \delta_2)\), which is equivalent to the optimal portfolio policy of the CRRA investor with \(\gamma = 1 - \frac{1}{1 - \delta_2}\). When \(\xi_t\) or \(\bar{\xi}_t\) goes to 0, \(\pi^*_t\) or \(\bar{\pi}_t\) goes to \(-\sigma^{-1} \kappa\), which is equivalent to the optimal portfolio policy of the CRRA investor with \(\gamma = \alpha\).

Fig. 5 shows that the consumption rate is increasing as a function of the wealth and the growth for a loss averse investor is slower than that for a CRRA investor. This is because a loss averse investor wishes to keep consuming more than the reference level for a long time. In addition, we can see that until \(X^*_t \geq 50\), the investor with a consumption constraint starts his portfolio and consumption. And \(\bar{X}_t = 50\). This illustrates our assumption \(\bar{X} > \bar{C}\) is entirely reasonable.

Fig. 6 reveals that a CRRA investor without consumption constraint always invests a constant fraction in stocks. While the fraction invested in stocks for a loss averse investor is \(V\)-shaped again as a function of wealth. When the investor does have a downside consumption constraint, the fraction invested in stocks exhibits concavity as the investor begins to insure
The fraction invested in stocks. The solid lines—the loss averse investor without a downside consumption constraint, the dashed lines—the loss averse investor with a downside consumption constraint, the dash-dot lines—the CRRA investor without a downside consumption constraint, the dotted lines—the CRRA investor with a downside consumption constraint.

The relation of optimal consumption rate and optimal wealth. The solid lines—the loss averse investor without a downside consumption constraint, the dashed lines—the loss averse investor with a downside consumption constraint, the dash-dot lines—the CRRA investor without a downside consumption constraint, the dotted lines—the CRRA investor with a downside consumption constraint.

himself. The fraction invested in stocks for a loss averse investor is less than that for a CRRA investor as he cares more about how much the consumption exceeds his reference level than consumption itself, stocks are less attractive. Moreover, The fraction for a loss averse investor with consumption constraint is the least because he must maintain a certain amount of consumption.
5. Conclusions

In this work, we discussed continuous-time optimal portfolio and consumption model in an infinite horizon. The closed-form solutions of the optimal portfolio and consumption policies were obtained for a loss averse investor with a downside consumption constraint. Finally, we compared the optimal consumption and portfolio policies for loss averse investor with the case of CRRA investor by some numerical results, which demonstrated the influence of loss aversion.

Our main conclusions are:

1. A loss averse investor likes consuming more money but exposing less to risk than a CRRA investor does.
2. A loss averse investor has two different kinds of consumption and portfolio behavior in the good states and the bad states of market.
3. The size of the good-states region is not only related to the initial wealth, the reference level, but also depends on the downside consumption constraint.
4. Whether there is a consumption constraint or not, the optimal portfolio policy of a loss averse investor goes to that of a CRRA investors when the state price density goes to the extreme cases (0 or $+\infty$).

However, in this paper, the investment horizon was assumed to be infinite. When a loss averse investor’s goal is maximizing the expected utility from both consumption and terminal wealth on a finite horizon, it is difficult to define both consumption and wealth reference level because they are closely correlated. But it is very significant and worth investigating, especially when the reference level depends on the past consumption or wealth. This would be discussed in our further work.

Acknowledgments

This work was Supported by National Natural Science Foundation of China [No.11471304, 11401556] and the fundamental research funds for the central universities [WK 2040000012].

References
