Fast Ambiguity Resolution for Pulsar-Based Navigation by Means of Hypothesis Testing

LIANGWEI HUANG
QINGQING LIN
XINYUAN ZHANG
PING SHUAI
Qian Xuesen Laboratory of Space Technology, China Academy of Space Technology, Beijing, China

The problem of the global navigation satellite system carrier-phase ambiguity resolution (AR) has drawn a lot of attention and to some extent been well solved in the past. However, the pulse-phase AR in the pulsar-based navigation has not been fully investigated due to its special feature that the source pulsars are far away enough to be considered stationary relative to the observers. In this paper, the indeterminate AR problem is proposed to describe this AR situation by a group of indeterminate measurement equations and a new method is developed that enables faster estimation of pulse-phase ambiguities. This method is based on the hypothesis testing that helps one to construct the ambiguity acceptance space at a certain significance level. The acceptance space is then reformulated as a linear form via singular value decomposition that is much easier to search for. Besides, the search algorithm is redesigned that uses particle swarm optimization to quickly find an initial solution and employs a new parameter to compress the search space. As a result, the ambiguity search can be performed much more efficiently especially at big problem sizes.

1. INTRODUCTION

Pulsars [1] are stably and rapidly rotating neutron stars that emit pulsed signals. Pulsar-based navigation (PNAV) [2], [3] is a revolutionary technology which uses X-ray pulsars as beacons and provides the autonomous navigation capability anywhere in the solar system. The position determination of a spacecraft depends on the cycled pulse-phase observation [4]. However, the number of integer phase cycles cannot be measured directly or it is ambiguous [5]. This ambiguity must be resolved so that the correct absolute position of a spacecraft in space can be determined. Thus, the ambiguity resolution (AR) problem is one of the most important points in PNAV [5], [6]. Especially, if the navigation system breaks down, it requires the correct ambiguity to reconstruct the initial positioning information when being restarted.

Before starting the discussion, let us survey the similar problem of the global navigation satellite system (GNSS) carrier-phase AR. Although GNSS AR is mainly used in precise relative positioning, its applications have grown to include attitude determination, formation flying, and other systems such as the field of interferometric synthetic aperture radar (InSAR) [7], [8]. The traditional GNSS AR includes three steps, which are as follows [9]:

1) The integer nature of the ambiguities is discarded and the so-called float solution is obtained.
2) The float solution is adjusted to yield the integer estimation.
3) All other parameters are corrected with the estimated integer ambiguities to obtain the so-called fixed solution.

The above process is known as the integer estimation and is now often considered solved [10]. Three best known integer estimators are the integer rounding, the integer bootstrapping, and the integer least-squares (ILS) [11]. The ILS is mechanized in the least-squares ambiguity decorrelation adjustment method ([12], [13]), which is popularly used because it considers the correlations amongst all ambiguities to improve the search speed.

The GNSS AR is not only the estimation, but also a decision on whether to accept the estimated integers [14]. Thus, the integer aperture (IA) estimation [11] is raised as a general framework for the ambiguity test. Under this framework, one can test the integer estimates by an aperture pull-in space generated from the preset failure rate [15]. Several such tests are the ratio test, the distance test, and the projector test [14]. However, the IA estimation is not designed to validate the underlying model [14]. In practice, the GNSS model is validated by the standard theory of hypothesis testing, which however is not quite applicable for the integer parameters [16]. Hence, a proper theory is still lacking and the GNSS integer-based model validation remains an open problem [10], [16].

In PNAV AR, the geometric structure of the observed sources is not varying with time unlike that of the navigation satellites because the pulsars are too far away. As a result, multiple epoch observations will not enhance the
ambiguity measurability. Furthermore, limited by the detection and time resources on orbit, one cannot expect to observe more pulsars to strengthen the model. Four necessary sources are used to form an AR unit, and extra sources are used only if AR is not completed or fails. Therefore, the PNAV AR is considering observing four sources for a single epoch, which is characterized by the linear four-dimensional indeterminate ambiguity measurement equations with an observation matrix that has a rank defect of three. The PNAV AR is thus stated as the indeterminate ambiguity resolution (IAR) problem in this paper.

For the rank-deficient situation, there exists the case that the integer estimation technique is used by the InSAR AR for deformation monitoring [8]. In this case, with an observation matrix that has a rank defect of one, the priori data are used to increase the rank or the constraint on the deformation rate is used in the ILS to generate reliable integer estimates [8]. So, the InSAR AR can be considered as an application of GNSS AR. However, to solve the PNAV IAR problem, one may not use the integer estimation methods of GNSS AR for two reasons: 1) The four-dimensional indeterminate measurement equations with a rank defect of three cannot yield a so-called float solution, and the priori data are too coarse when the navigation system is restarted. 2) Even one-cycle bias of ambiguity will cause great position error in PNAV, so ambiguities should be precisely resolved and the ILS solution is not quite trusted because real ambiguities might be excluded when the measurement error occurs.

The performance of the PNAV AR can be evaluated in two aspects, which are the success probability and the time efficiency. For the first aspect, it is the success probability not the AR accuracy that is concerned because even one-cycle error is seen as a failure. In this case, instead of integer estimation, the idea of integer-based model validation can be referred to. Unlike GNSS, the four-dimensional indeterminate characterization of IAR makes it possible to build a hypothesis testing validation model to fully evaluate all ambiguity candidates and to control the success probability. For the second aspect, if the success probability is set to a certain level, the least AR time cost is expected. To our knowledge, there has not been such a statistical model for the IAR problem presented in the literature, so that the ambiguity search algorithms designed by earlier researchers cannot be evaluated in both aspects. In [5], the ambiguity candidates were tested for the search space defined by the measurement accuracy. A basic search structure was constructed in [6]: Three navigation pulsars and one test pulsar constitute an AR unit, and the test pulsar is replaced one after another until a unique ambiguity is found for three navigation pulsars. Recently, several authors proposed their algorithms to accelerate the search process. In [17], a linear-type test equation was obtained to reduce the time cost, and in [18], the matching search pattern was used to further decrease the computational complexity. These algorithms cannot tell the exact distribution of the test quantities and their test thresholds are selected empirically, so it is hard to balance the efficiency and the success probability during the AR process.

The purpose of this paper is to enhance the PNAV AR performance in both success probability and time efficiency. The hypothesis testing is used to decide the acceptance space and the singular value decomposition (SVD) is applied to obtain the linear-form acceptance space (LAS) model. Then, the methods in [17] and [18] are reorganized as two algorithms to search for the LAS. A new search algorithm based on matching search technique is proposed as the particle swarm optimization (PSO) and compressed-pattern matching search cascade (PSO-CPMS) algorithm, which introduces a new parameter to compress the search pattern and uses the PSO to quickly find an initial ambiguity solution. The numerical simulation is designed to investigate the success probability and the time costs at different problem sizes for the three search algorithms. Positive results are finally obtained to validate the effectiveness and the efficiency of the newly proposed algorithm.

II. NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Set containing currently accepted ambiguities.</td>
</tr>
<tr>
<td>b&lt;sub&gt;i&lt;/sub&gt;</td>
<td>Line-of-sight vector of the i&lt;sup&gt;th&lt;/sup&gt; pulsar.</td>
</tr>
<tr>
<td>B</td>
<td>Measurement matrix.</td>
</tr>
<tr>
<td>C</td>
<td>Vacuum speed of light.</td>
</tr>
<tr>
<td>C</td>
<td>Matrix defined by ( C = I - BH ).</td>
</tr>
<tr>
<td>D</td>
<td>LAS model parameter.</td>
</tr>
<tr>
<td>e</td>
<td>Measurement error vector.</td>
</tr>
<tr>
<td>f&lt;sub&gt;i&lt;/sub&gt;</td>
<td>i&lt;sup&gt;th&lt;/sup&gt; order time derivative of the pulse frequency.</td>
</tr>
<tr>
<td>H</td>
<td>Weighted {1,3}-inverse of B.</td>
</tr>
<tr>
<td>I</td>
<td>Identity matrix.</td>
</tr>
<tr>
<td>k</td>
<td>Vector defined by ( k = Wu_1 ).</td>
</tr>
<tr>
<td>l</td>
<td>Vector defined by ( l = \sigma_1v_1^T ).</td>
</tr>
<tr>
<td>m</td>
<td>Ambiguity vector.</td>
</tr>
<tr>
<td>m&lt;sub&gt;i&lt;/sub&gt;</td>
<td>Ambiguity value of the i&lt;sup&gt;th&lt;/sup&gt; pulsar.</td>
</tr>
<tr>
<td>n</td>
<td>Pulsar’s normalized position vector.</td>
</tr>
<tr>
<td>P&lt;sub&gt;L&lt;/sub&gt;</td>
<td>Lower boundary of the success probability.</td>
</tr>
<tr>
<td>P&lt;sub&gt;suc&lt;/sub&gt;</td>
<td>Success probability.</td>
</tr>
<tr>
<td>R</td>
<td>Error variance matrix.</td>
</tr>
<tr>
<td>S</td>
<td>Real number set.</td>
</tr>
<tr>
<td>S&lt;sub&gt;A&lt;/sub&gt;</td>
<td>Ambiguity set.</td>
</tr>
<tr>
<td>S&lt;sub&gt;1&lt;/sub&gt;</td>
<td>Matching search pattern set.</td>
</tr>
<tr>
<td>S&lt;sub&gt;2&lt;/sub&gt;</td>
<td>Compressed matching search pattern set.</td>
</tr>
<tr>
<td>S&lt;sub&gt;p&lt;/sub&gt;</td>
<td>Problem size.</td>
</tr>
<tr>
<td>T</td>
<td>Pulsar’s proper time.</td>
</tr>
<tr>
<td>T</td>
<td>Test statistic.</td>
</tr>
<tr>
<td>u&lt;sub&gt;1&lt;/sub&gt;</td>
<td>C’s left singular vector corresponding to ( \sigma_1 ).</td>
</tr>
<tr>
<td>U</td>
<td>Ambiguity test interval.</td>
</tr>
<tr>
<td>U&lt;sub&gt;1&lt;/sub&gt;</td>
<td>Initial ambiguity test interval.</td>
</tr>
<tr>
<td>v&lt;sub&gt;1&lt;/sub&gt;</td>
<td>C’s right singular vector corresponding to ( \sigma_1 ).</td>
</tr>
<tr>
<td>V&lt;sub&gt;1&lt;/sub&gt;</td>
<td>Matching search pattern.</td>
</tr>
<tr>
<td>V&lt;sub&gt;2&lt;/sub&gt;</td>
<td>Compressed matching search pattern.</td>
</tr>
<tr>
<td>W</td>
<td>Square root matrix of ( R^{-1} ).</td>
</tr>
<tr>
<td>X</td>
<td>Spacecraft’s positional vector.</td>
</tr>
<tr>
<td>y</td>
<td>Measurement vector.</td>
</tr>
</tbody>
</table>
\[ z \] Transformed measurement vector.
\[ Z \] Integer set.
\[ \alpha \] Significance level.
\[ \alpha_{P} \] Pulsar’s right ascension.
\[ \delta m \] Ambiguity increment.
\[ \delta \] Test interval shift.
\[ \delta_{T} \] Normalized test interval shift.
\[ \delta_{P} \] Pulsar’s declination.
\[ \Delta \] Time delay of the pulse signal.
\[ \epsilon_{L} \] Phase linearization error.
\[ \epsilon_{W} \] Phase measurement error.
\[ \phi \] Fractional part of the pulse phase.
\[ \Phi \] Pulse phase.
\[ \gamma \] Matching search pattern compression parameter.
\[ \theta_{L} \] Lower boundary of the \( i \) th pulsar’s ambiguity.
\[ \theta_{U} \] Upper boundary of the \( i \) th pulsar’s ambiguity.
\[ \rho \] Spacecraft’s position vector.
\[ \rho_{E} \] Geocenter position vector.
\[ \rho_{m} \] Maximum distance.
\[ \sigma_{1} \] \( C \)'s nonzero singular value.
\[ \tau \] Spacecraft’s proper time.
\[ \emptyset \] Null set.
\[ | \cdot | \] Absolute value or element number of a set.
\[ || \cdot || \] Two-norm.
\[ || \cdot ||_{R}^{-1} \] \( R^{-1} \)-norm.

Subscripts
\[ 0 \] Initial.
\[ n \] Of three navigation pulsars.

Superscripts
\[ E \] Of the Earth.
\[ P \] Of the pulsar.
\[ X \] Of the spacecraft.
\[ \hat{\cdot} \] Measured.

III. PRELIMINARIES AND PROBLEM STATEMENT

A. Overview of PNAV

The procedure that can be followed for PNAV is divided into five steps. In the first step, photons from pulsars are detected and recorded at spacecraft, which yields a series of photon time-of-arrivals (TOAs). In the second step, one avails the rotational periodicity to epoch-fold the photon TOAs and obtains the measured pulse profile of the observed pulsar. The measured pulse profile is cross-correlated with the standard pulse profile to measure the pulse TOA, which tells the pulse phase on spacecraft at a given epoch [19]. The third step copes with the Doppler effect caused by the spacecraft’s motion on orbit and the pulsar’s own motion if it is in a binary system. The segmented measured pulse phases are tracked to extract one precise pulse-phase measurement that directly contains the positional information [4], [20].

The fourth step is the AR process concerned in this paper. The measured pulse phase is only the fractional part. The unknown integer part is called the pulse-phase cycle ambiguity, which should be resolved to correctly determine the spacecraft position. On condition that three positional components are unknown, at least four pulsars are needed to resolve the ambiguity by the integer-based technique. Limited by the resource on orbit, one cannot expect to observe more pulsars for AR. Thus, an AR unit is often constituted by four pulsars that are three navigation pulsars and one test pulsar. Extra test pulsars are added one after another until a unique ambiguity is found for three navigation pulsars. The four pulsars had better be observed simultaneously. If only one detector is equipped, different pulsars can be observed sequentially. Measurements are then synchronized to the same epoch according to the method in [21]. Once the ambiguity is resolved, the last step incorporates the phase measurements with the orbit dynamics by the filtering technique and outputs the best estimated position, velocity, and time of the spacecraft [22], [23].

B. Ambiguity Measurement Formulation

Next, the formulation of PNAV ambiguity measurement is investigated. Suppose it takes total time \( \Delta_{\text{total}} \) for the pulse signal to be transmitted from the pulsar to the spacecraft. This course can be comprehended as the transmission of the same pulse phase from the pulsar to the spacecraft. The transmission time delay satisfies \( \tau - \Delta_{\text{total}} = t \), where \( \tau \) is the spacecraft proper time of the pulse arrival and \( t \) is the pulsar proper time of the pulse emission. Thus, the observed pulse phase is the delayed inherent phase:

\[ \Phi^{X}(\tau) = \Phi^{I}(\tau - \Delta_{\text{total}}) \]

where \( \Phi^{X} \) denotes the observed phase on spacecraft, \( \Phi^{I} \) is the pulsar’s inherent rotation phase. The inherent phase \( \Phi^{I} \) is expressed by [24], [25]

\[ \Phi^{I}(t) = \Phi^{I}_{0} + f_{0}(dt) + 1/2 f_{1}(dr)^{2} + 1/6 f_{2}(dr)^{3} \]

where \( t \) is the pulsar proper time, \( \Phi^{I}_{0} \) is the initial phase evaluated at \( t = E_{\text{FRQ}} \) with \( E_{\text{FRQ}} \) being the frequency epoch, \( f_{0} \) is the rotation frequency, \( f_{1} \) and \( f_{2} \) are respectively the first-order and the second-order time derivative of the rotation frequency, and \( d \) denotes the time elapse with respect to the frequency epoch, i.e., \( dt = t - E_{\text{FRQ}} \).

The measured value of \( \Phi^{X} \) contains only the fractional part, noted by

\[ \hat{\Phi}^{X} = \Phi^{I}(\tau - \Delta_{\text{total}}) - m^{X} + \epsilon_{W} \]

where \( m^{X} \) is the absolute ambiguity that is the unknown integer part and \( \epsilon_{W} \) denotes the phase measurement error. The total time delay \( \Delta_{\text{total}} \) contains the solar system delay, the interstellar space delay, and the binary system delay (if the pulsar is in a binary system) [25]. Consider that PNAV uses the isolated X-ray pulsars in the outer space. Thus, the binary system delay is zero, the delay related to the atmosphere is omitted, and the dispersion delay is neglected. Then, according to [21], the total time delay is formulated to be \( \Delta_{\text{total}} = \Delta_{rs0} + \Delta_{ex} \), where \( \Delta_{rs0} \) is the basic Roemer delay and \( \Delta_{ex} \) is the extended delay containing much smaller items ([21], [25], [26]). The basic Roemer delay can be expressed by \( \Delta_{rs0} = -c^{-1} n^{T} \rho \), where \( c \) is the vacuum speed of light, \( \rho \) is the spacecraft’s position vector in the barycentric celestial reference system (BCRS), and \( n \) denotes the
normalized position vector of the pulsar in the BCRS ([25]):

\[ n = \left[ \cos \delta_p \cos \alpha_p, \cos \delta_p \sin \alpha_p, \sin \delta_p \right]^T \]

where \( \alpha_p \) and \( \delta_p \) are respectively the right ascension and the declination of the pulsar.

The absolute ambiguity \( m^X \) is a large number because the frequency epoch is always old time. Besides, the measurement equation (1) is nonlinear with respect to the spacecraft position, so it is less manageable to resolve \( m^X \) directly. Fortunately, the nonlinear equation (1) can be linearized by differentiating from a reference point, e.g., the Earth center. Define the reference phase to be \( \phi^E - \phi^X = -c^{-1} f_i n^T (\rho - \rho_E) + m^X - m^E + \varepsilon_L - \varepsilon_W \) (2)

where \( \varepsilon_L \) is the linearization error, and \( f_i = f_0 + f_1 (d\tau) + 1/2 f_2 (d\tau)^2 \). The linearization error analysis could refer to [21]. The linear equation is obtained as (2), which can be rewritten to be

\[ y = m + b^T x + e \] (3)

where \( y = \phi^E - \phi^X \) is the phase difference measurement, \( m = m^X - m^E \) denotes the difference ambiguity that is a much smaller number, \( b = -c^{-1} f_i n \) is the line-of-sight vector, \( x = \rho - \rho_E \) is the unknown position vector, and \( e = \varepsilon_L - \varepsilon_W \) represents the measurement error. The geometry of the PNAV AR is shown in Fig. 1, which illustrates the relationship among the absolute ambiguity, the reference ambiguity, and the reference position. It is the difference ambiguity that is intended to be resolved by PNAV AR. In the following text, the term ambiguity denotes difference ambiguity if there is no special instruction.

C. Problem Statement

According to (3), the vectorial model of the linear ambiguity measurement for an IAR unit can be formulated as

\[ y = m + B x + e \] (4)

Here, \( m := [m^T_1, m^T_4] \in \mathbb{Z}^3 \times \mathbb{Z} \) is the integer ambiguity vector of an IAR unit, where \( m_n = [m_1, m_2, m_3]^T \in \mathbb{Z}^3 \) is the three navigation pulsars’ ambiguity vector and \( m_4 \in \mathbb{Z} \) is the test pulsar’s ambiguity. \( x := [x_1, x_2, x_3]^T \in \mathbb{R}^3 \) represents the spacecraft’s position vector relative to the origin that is chosen to the geocenter if the mission is near the Earth or the solar system barycenter if the mission is deep space. \( B := [b_1, b_2, b_3, b_4] \in \mathbb{R}^{4 \times 3} \) is the measurement matrix decided by the line-of-sight vector \( b_i \) (\( i = 1, 2, 3, 4 \)). \( y := [y_1, y_2, y_3, y_4]^T \in \mathbb{R}^4 \) represents the measurement vector satisfying \( 0 \leq y_i < 1 \) (\( i = 1, 2, 3, 4 \)). \( e \in \mathbb{R}^4 \) is the measurement error vector.

Assumption 1 \( B \) has full column rank: \( \text{rank}(B) = 3 \).

Assumption 2 Assume \( e \) to follow the zero-mean four-dimensional normal distribution: \( e \sim N_d(0, R) \), where \( R \) is the symmetric positive definite error variance matrix satisfying \( E(ee^T) = R > 0 \).

The PNAV IAR problem is how to decide the ambiguity vector \( m \) and the spacecraft position vector \( x \) from the model defined by (4). In the IAR process, the three navigation pulsars are always fixed, and the test pulsar can be replaced one after another if \( m_n \) cannot be uniquely defined.
Commonly, the spacecraft is confined in a coarse region such as the solar system or the near-Earth space. Let \( \rho_m \) denote the spacecraft’s possible maximum distance from the origin (see Fig. 1). There exists \(|x| < \rho_m\), where \( || \cdot || \) represents the two-norm of a vector. Then, the ambiguity boundaries and the problem size can be defined.

**Definition 1** The ambiguity boundaries are defined by the lower and upper boundaries of the ambiguity elements, i.e., \( \hat{\theta}_{UL} \leq \theta_{ij} \leq \hat{\theta}_{UL} \), where \( \hat{\theta}_{UL} \) is the lower boundary satisfying \( \hat{\theta}_{UL} = \text{round}(-||b_i||/\rho_m) \) and \( \hat{\theta}_{UL} \) is the upper boundary satisfying \( \hat{\theta}_{UL} = \text{round}((||b_i||/\rho_m) + 1) \).

**Definition 2** The problem size \( S_p \) is defined by the number of ambiguity candidates within the boundaries for the IAR unit consisting of three navigation pulsars and the first test pulsar. \( S_p \) increases with \( \rho_m \) and can be calculated by

\[
S_p = \prod_{i=1}^{4} (\hat{\theta}_{UL} - \hat{\theta}_{UL} + 1).
\]

**IV. MODEL CONSTRUCTION**

If the ambiguity estimation \( \hat{m} \) is specified, the position estimation \( \hat{x} \) can be decided by weighted least square (WLS) from (4). Let \( \hat{e} = y - \hat{m} - B \hat{e} \) denote the measurement residual. The WLS minimizes the \( R^{-1} \)-norm of \( \hat{e} \) that is expressed by

\[
\| \hat{e} \|_{R^{-1}} = (\hat{e}^T R^{-1} \hat{e})^{1/2}.
\]

The solution to the WLS is \( \hat{x} = H(y - \hat{m}) \), where \( H \) represents \( B \)'s weighted \( \{1,3\} \)-inverse satisfying

\[
\begin{bmatrix}
BHB &=& B \\
(R^{-1}BH)^T &=& R^{-1}BH
\end{bmatrix}.
\]

From Assumption 1, \( H \) is unique and can be expressed by

\[
H = (B^T R^{-1} B)^{-1} B^T R^{-1}.
\]

Let \( C = I - BHB = I - B(B^T R^{-1} B)^{-1} B^T R^{-1} \). \( \hat{e} \) can be rewritten to be \( \hat{e} = C(e + m + Bx - \hat{m}) \). From (4), there exists \( \hat{e} = C(e + m + Bx - \hat{m}) \). Because \( CB = B - BHB = 0 \), \( \hat{e} \) is then rewritten as

\[
\hat{e} = Ce + C(m - \hat{m}).
\]

**Lemma 1** The matrix \( C \) has the following properties: 1) \( C \) is idempotent; 2) \( C \) has unique nonzero singular value and \( \text{rank}(C) = 1 \).

**Proof** Because \( H \) is \( B \)'s weighted \( \{1,3\} \)-inverse, there exist \( HB = I \) and \( (BH)(BH) = BH \) (i.e., \( BH \) is idempotent). \( BH \) shares the same nonzero eigenvalues with \( HB \), so \( BH \)'s eigenvalues must be \( (1, 1, 1, 0) \) and there is \( \text{rank}(BH) = 3 \). Then, \( C \) is also idempotent, and there is \( \text{rank}(C) = 4 - \text{rank}(BH) = 1 \). Therefore, \( C \) has unique nonzero singular value.

**Remark 1** From Lemma 1, \( C \) can be factorized to be \( C = u_1 \sigma_1 v_1^T \) via SVD, where \( \sigma_1 \) is \( C \)'s nonzero singular value, and \( u_1 \) and \( v_1 \) are respectively \( C \)'s left singular vector and right singular vector corresponding to \( \sigma_1 \).

**Lemma 2** On condition that \( q \) follows the \( q \)-dimensional normal distribution \( q \sim N_q(0, \Sigma) \), where \( \Sigma > 0 \) and the given matrix \( A \) satisfies \( A^T = A \) and \( \text{rank}(A) = r \), a necessary and sufficient condition for \( q^T Aq \) to follow the chi-squared distribution with \( r \) degrees of freedom \( \chi^2(r) \) is \( \Sigma A \Sigma A = \Sigma A \Sigma \).

The hypothesis testing is used to distinguish between true and false ambiguities. The null hypothesis and the alternative hypothesis are respectively

\[
H_0: \hat{m} = m, \quad H_1: \hat{m} \neq m.
\]

The test statistic is designed as the quadratic form in the residual:

\[
T = \hat{e}^T R^{-1} \hat{e}.
\]

At \( H_0 \), from (6), there is \( \hat{e} = Ce \), so \( T \) becomes a quadratic form in \( e \):

\[
T = e^T C^T R^{-1} C e.
\]

**Theorem 1** \( T \) is distributed according to the chi-squared distribution with one degree of freedom: \( T \sim \chi^2(1) \).

**Proof** Under Assumption 2, consider \( T \) in (7). Due to \( R > 0, R^{-1} > 0 \), and with \( \text{rank}(C) = 1 \) (see Lemma 1), the following relationship holds:

\[
1 \leq \text{rank}(C^T R^{-1} C) \leq \text{rank}(R^{-1} C) = \text{rank}(C) = 1.
\]

Therefore,

\[
\text{rank}(C^T R^{-1} C) = 1.
\]

From (5), there exists

\[
C^T R^{-1} = R^{-1} - (R^{-1} BX)^T = R^{-1} - (R^{-1} BX) = R^{-1} C.
\]

According to the idempotence of \( C \) (see Lemma 1) and (9), the following expression is obtained:

\[
R(C^T R^{-1} C) R(C^T R^{-1} C) R = R(C^T R^{-1} C) R.
\]

With the two conditions respectively given in (8) and (10), from Lemma 2, \( T \sim \chi^2(1) \).

Under Theorem 1, given the significance level \( \alpha \), the acceptance region for \( \hat{m} = m \) is \( T \leq \chi^2(1, \alpha) \), i.e.,

\[
\| C(y - \hat{m}) \|_{R^{-1/2}}^2 \leq \chi^2(1, \alpha).
\]

Equation (11) defines a region quite probably containing the true ambiguity, which we call “acceptance space” (AS) model. However, the AS model is nonlinear with respect to \( \hat{m} \), which is inconvenient for the search algorithm design. The equivalent linear form is then required.

Because \( R^{-1} > 0 \), a unique matrix \( W \geq 0 \) can be derived satisfying \( W^2 = R^{-1} \). Thus, according to the SVD \( C = u_1 \sigma_1 v_1^T \) in Remark 1, the left-hand side of (8) is expressed as

\[
\| C(y - \hat{m}) \|_{R^{-1}}^2 = \| z - km_L \|_2^2
\]
where \( z = WCy, \ k = Wu_1, \) and \( m_L = l^T \hat{m} \) with \( l := [l_1, l_2, l_3, l_4]^T \in \mathbb{R}^4 \) satisfying \( l = \sigma_l v^T \). Furthermore, there is \( z = (\sigma_l v^T y)(Wu_1) = (\sigma_l v^T y)k, \) which indicates that \( z \) is parallel to \( k \), so
\[
(z^T k)^2 = ||z||^2 ||k||^2. \tag{13}
\]

From (12), the AS model (11) can be rewritten to be
\[
||z - km_L||^2 \leq \chi^2 (1). \tag{14}
\]

The left-hand side of (14) is a parabolic function with respect to the independent variable \( m_L \), thus, with (13), (14) can be resolved:
\[
\hat{M} - D \leq m_L \leq \hat{M} + D \tag{15}
\]
where \( \hat{M} = z^T k ||k||^{-2}, \ D = ||k||^{-1} \sqrt{\chi^2 (1)}. \)
Rewrite (15) to be
\[
[l^T \hat{m} - \hat{M}] \leq D \tag{16}
\]
which is a necessary and sufficient condition for the AS model (11). We finally obtain the LAS model as is shown by (16).

**Remark 2** In one test operation, according to (11), the probability of rejecting the true ambiguity is \( \alpha \). However, we only have to find the right \( m_a \) of the three navigation pulsars even if at a false \( \hat{m}_a \), so the probability of rejecting \( m_a \) will be less than \( \alpha \). Then, if only one \( \hat{m}_a \) remains after implementing \( N \) test pulsars, the probability of finding the true one will be greater than \( (1 - \alpha)^N \). This value serves as a lower boundary of the success probability so that one can control the success probability by adjusting the \( \alpha \) value.

**V. SEARCH ALGORITHM DESIGN**

**A. Exhaustive Search**

Under the LAS model, the task for IAR is to solve (16). The simplest way that can be imagined is the exhaustive search (ES) algorithm. In terms of efficiency and search structure, the ES algorithm is similar to the one in [17]. It can be interpreted as follows.

According to Definition 1, the search space can be defined by lower and upper boundaries of the ambiguity, i.e., for \( i = 1, 2, 3, 4, m_i \in [\theta_L, \theta_U] \). Let us assume without loss of generality that \( l_4 > 0 \) and define the interval \( U \) as
\[
U = [(\hat{M} - D - l_4^T \hat{m}_a)/l_4, \ (\hat{M} + D - l_4^T \hat{m}_a)/l_4] \tag{17}
\]
where \( l_4 = [l_1, l_2, l_3]^T \). Let \( S \) be the set containing all possible navigation pulsars’ ambiguities within their boundaries:
\[
S = \{ \hat{m}_a \mid \hat{m}_i \in [\theta_L, \theta_U], \ \hat{m}_i \in \mathbb{Z}, \ i = 1, 2, 3 \}. \tag{18}
\]

Let \( A \) represent the set containing currently accepted navigation pulsars’ ambiguities, let \( |\cdot| \) denote the element number of a set, and let \( j \) be the sequence number of the test pulsar. Then, the ES algorithm is expressed as follows.

**Algorithm 1:** Exhaustive search (ES).

**Step 1:** For the \( j \)th (started with \( j = 1 \)) test pulsar, use three-dimensional array traverse to construct the set \( A \) as
\[
A = \{ \hat{m}_a \in S \mid U \cap [\theta_L, \theta_U] \cap \mathbb{Z} \neq \emptyset \}.
\]

**Step 2:** If \( |A| = 1 \), the search is fulfilled and the remaining element in \( A \) is the navigation pulsars’ ambiguity. If \( |A| = 0 \), the search fails and aborts. If \( |A| > 1 \), set \( j = j + 1 \) and return to Step 1.

**Remark 3** Although the ES algorithm’s time cost increases greatly as the problem size scales up, it is easy to carry out and is efficient at a small problem sizes. If more than one test pulsar is required, after implementation of the first test pulsar, the ES can be used in the test operations of the second and the succeeding test pulsars.

**B. Matching Search**

If an initial solution \( \hat{m}_0 = [\hat{m}_{01}, \hat{m}_{02}, \hat{m}_{03}, \hat{m}_{04}] \) is obtained satisfying (16), \( \hat{m}_{04} \) must be within the interval \( U_1 \) expressed by (still assuming \( l_4 > 0 \))
\[
U_1 = [(\hat{M} - D - l_4^T \hat{m}_{04})/l_4, \ (\hat{M} + D - l_4^T \hat{m}_{04})/l_4] \tag{19}
\]
where \( \hat{m}_{04} = [\hat{m}_{01}, \hat{m}_{02}, \hat{m}_{03}] \).

For an accepted \( \hat{m}_a \), the interval \( U \) defined by (17) must contain at least one integer. \( U \) brings on a shift from \( U_1 \) by \( \delta := -1/l_4 l_4^T (\hat{m}_a - \hat{m}_{04}) \) in the positive direction, so \( U \) and \( U_1 \) have the same length \( 2D/l_4 \). The shift \( \delta \) can be identified with \( \delta : = \delta - \text{round}(\delta) \in [-0.5, 0.5], \) which makes no difference in judging whether \( U \) contains an integer. If \( 2D/l_4 \geq 0.5 \), there is naturally \( |\delta| \leq 2D/l_4 \). If \( 2D/l_4 < 0.5 \), \( \delta \) should also satisfy \( |\delta| \leq 2D/l_4 \) to ensure \( U \) contains an integer. Therefore, a necessary condition for \( \hat{m}_a \) to be accepted is
\[
|\delta| \leq 2D/l_4. \tag{20}
\]

Constitute the set \( V_1 \) that is called the matching search pattern [18]:
\[
V_1 = \left\{ \hat{m}_a \mid \mid \delta \hat{m}_a \mid \leq 2D/l_4, \ |\delta \hat{m}_n| < \theta_{L} - \theta_{U}, \ \delta \hat{m}_i \in \mathbb{Z}, \ i = 1, 2, 3 \right\}.
\]

The set \( V_1 \) depends on \( D, l, \theta_L, \) and \( \theta_U \). These variables further depend on the measurement matrix \( B \) and the error variance matrix \( R \) of the pulsars in the IAR unit, the significance level \( \alpha \), and the maximum range \( \rho_{m} \). These quantities will not vary with time. Before mission launch, \( \rho_{m} \) can be determined from the mission type, \( \alpha \) can be set according to the AR success probability control demand, navigation and test pulsars can be preselected so that \( B \) and \( R \) are known parameters. Thus, the matching search pattern that tells the feasible ambiguity increments can be determined before launch and stored in the on-board computer. Then, on orbit, it is only required to find any \( \hat{m}_{04} \) and test whether the ambiguity candidate \( \hat{m}_{04} + \delta \hat{m}_a \) is within its boundaries. After implementation of the first test pulsar, the probable
ambiguities of the navigation pulsars will be confined to a small region, so the ES algorithm can be used to obtain a unique solution for the next test pulsars (see Remark 3). Thus, using the LAS model, we reorganize the algorithm in [18] to form the sequential search and matching search cascade (SS-MS) algorithm as follows.

**Algorithm 2:** Sequential search and matching search cascade (SS-MS).

**Step 1:** For the jth \((j = 1)\) test pulsar, sequentially search \(S\) in (18) for an \(\hat{m}_{a0}\) that satisfies \(U_1 \cap [\theta_L, \theta_U] \cap \mathbb{Z} \neq \emptyset\).

**Step 2:** Constitute the set \(S_1:\)
\[
S_1 = \{ \hat{m}_n = \hat{m}_{a0} + \delta \hat{m}_n \mid \delta \hat{m}_n \in V_1, \hat{m}_n \in [\theta_L, \theta_U], i = 1, 2, 3 \}.
\]

**Step 3:** If \(|S_1| = 1\), the search is fulfilled and the remaining element in \(S_1\) is the navigation pulsars’ ambiguity. If \(|S_1| = 0\), the search fails and aborts. If \(|S_1| > 1\), set \(j = j + 1\), go to Step \(1\) of Algorithm \(1\), and continue with the steps of Algorithm \(1\).

C. Compressed-Pattern Matching Search

To further improve the efficiency of the matching search, we use a new parameter \(\gamma\) to compress the matching search pattern. If an initial solution \(\hat{m}_0\) is obtained satisfying
\[
|I^T \hat{m}_0 - \hat{M}| \leq D/\gamma \tag{21}
\]
where \(\gamma \geq 1\), \(\hat{m}_{04}\) must be within the interval \(U_2\) expressed by (still assuming \(l_4 > 0\))
\[
U_2 = [(\hat{M} - D/\gamma - I^T \hat{m}_{a0})/l_4, (\hat{M} + D/\gamma - I^T \hat{m}_{a0})/l_4].
\]
(22)

Now, the length of \(U_2\) is \(2D/\gamma |l_4|\), and \(U\) brings on not only a shift from \(U_2\) by \(\delta\) (or equivalently \(\delta_i\)) in the positive direction but also an expansion by \((1 - 1/\gamma)D/|l_4|\) at both sides of \(U_2\). Then, if an arbitrary \(\hat{m}_n\) is accepted according to (16), which indicates that the correspondent interval \(U\) contains at least one integer, there must exist
\[
|\delta_i| \leq 2D/\gamma |l_4| + (1 - 1/\gamma)D/|l_4| = (1 + 1/\gamma)D/|l_4|. \tag{23}
\]

Thus, a new necessary condition for \(\hat{m}_n\) to be accepted is
\[
|\delta_i| \leq (1 + 1/\gamma)D/|l_4|. \tag{24}
\]

According to (24), if \(\gamma > 1\), a compressed matching search pattern \(V_2\) can be built:
\[
V_2 = \\{ \hat{m}_{a0} \mid |\delta \hat{m}_{a0}| \leq (1 + 1/\gamma)D/|l_4|, |\delta \hat{m}_i| < \theta_U - \theta_L, \delta \hat{m}_i \in \mathbb{Z}, i = 1, 2, 3 \}.
\]

Compared to (16), it is harder to search for an initial solution for (21), so we introduce the PSO method [27], [28], hoping to find \(\hat{m}_{a0}\) faster. We follow [29] to deal with the integer constraint, which uses the step moving strategy instead of rounding the particle velocity. The difference is that in this paper the particles’ positions are initialized to be scattered near the hyperplane \(|I^T \hat{m} - \hat{M}| = 0\) according to the cost function \(f(\hat{m}) = |I^T \hat{m} - \hat{M}|\).

Let \(S\) denote the number of particles in the swarm, let \(N_t\) denote the maximum number of iterations, let \(U\) represent the uniform distribution, let \(g = [g_1, g_2, g_3, g_4]^T\) be the swarm’s best known position, let \(p_k = [p_{k1}, p_{k2}, p_{k3}, p_{k4}]^T\) be the best known position of the \(k\)th particle, and let \(\hat{m}_k = [\hat{m}_{k1}, \hat{m}_{k2}, \hat{m}_{k3}, \hat{m}_{k4}]^T\) denote the current position of the \(k\)th particle. Then, the PSO is used as follows.

**Algorithm 3:** Integer PSO algorithm searching for an initial solution for (21).

**Step 1:** For each particle \(k = 1, 2, \ldots, S\), do:

1) For \(i = 1, 2, 3\), pick \(r_{ki} \sim U(0,1)\) and let \(\hat{m}_{ki} = \text{round}(r_{ki})\).
2) Let \(m_{ki} = \text{round}[(\hat{M} - I^T \hat{m}_{a0})/l_4]\).
3) Initialize the particle’s best known position to its initial position: \(p_k \leftarrow \hat{m}_k\).
4) If \(f(p_k) < f(g)\), update \(g: g \leftarrow p_k\).

**Step 2:** Until \(f(g) < D/\gamma\) or iteration times reach \(N_t\), for each particle \(k = 1, 2, \ldots, S\), do:

a) For \(i = 1, 2, 3, 4\), do:

   a.1) Pick \(r_p \sim U(0,1)\), let \(\hat{M}_{ki} \sim U(0,1)\), and \(r_d \sim U(0,1)\). \(r_d \sim U(0,1)\).
   a.2) Let \(v_{ki} = c_p r_p (\hat{m}_{ki} - \hat{m}_{ki}) + c_g r_g (g_i - \hat{m}_{ki})\), where \(c_p = c_g = 2\).
   a.3) Update the particle’s position:
   \[
   \hat{m}_{ki} \leftarrow \hat{m}_{ki} + v_{ki}.
   \]
   where
   \[
   d = \begin{cases} 
   1, & r_d \leq 1/3 \\
   0, & 1/3 < r_d \leq 2/3 \\
   -1, & r_d > 2/3 
   \end{cases}
   \]

b) If \(f(\hat{m}_k) < f(p_k)\), update \(p_k: p_k \leftarrow \hat{m}_k\).

c) If \(f(p_k) < f(g)\), update \(g: g \leftarrow p_k\).

**Step 3:** The initial solution for (21) is found: \(\hat{m}_0 = g\).

Then, our newly developed ambiguity search algorithm is named PSO-CPMS algorithm as follows.

VI. SIMULATION EXAMPLE AND RESULTS

The PSO-CPMS algorithm is compared with the ES algorithm and the SS-MS algorithm in this section. They are simulated at different problem sizes. The isolated pulsars
Algorithm 4: PSO and PSO-CPMS.

Step 1: For the jth \((j = 1)\) test pulsar, use Algorithm 3 to find an initial solution \(\hat{\mathbf{m}}_0\) for (21) and determine \(\hat{\mathbf{m}}_{n0}\) from \(\hat{\mathbf{m}}_0\).

Step 2: Constitute the set \(S_2:\)

\[
S_2 = \{ \hat{\mathbf{m}}_n = \hat{\mathbf{m}}_{n0} + \delta \hat{\mathbf{m}}_n \mid \delta \hat{\mathbf{m}}_n \in \mathbf{V}_2, \hat{\mathbf{m}}_i \in \{\theta_{li}, \theta_{ui}\}, i = 1, 2, 3\}.
\]

Step 3: If \(|S_2| = 1\), AR is fulfilled and the remaining element in \(S_2\) is navigation pulsars’ ambiguity. If \(|S_2| = 0\), the search fails and aborts. If \(|S_2| > 1\), set \(j = j + 1\), go to Step 1 of Algorithm 1 and continue with the steps of Algorithm 1.

### Table I

Matching Search Pattern Sizes at Different Problem Sizes

| \(\rho_m / (10^5 \text{ m})\) | \(S_p\) | \(|V_1|\) | \(|V_2|\) |
|----------------|-------|-------|-------|
| 5              | 384   | 65    | 49    |
| 10             | 1152  | 169   | 109   |
| 15             | 5760  | 621   | 387   |
| 30             | 45 056 | 2843  | 1661  |
| 50             | 254 592 | 10 889 | 6113  |
| 100            | 3 440 800 | 76 525 | 43 049 |
| 200            | 50 652 000 | 584 033 | 328 529 |

By the time cost of ES to exceed the time cost of AR can be done by one test pulsar and there is \(P_{\text{ss}} \approx P_L\).

Fig. 2. Ambiguity resolution time cost versus the problem size plotted in logarithmic scale.

With the simulation computer’s CPU working at 2.2 GHz, the average AR time costs in seconds versus different problem sizes are plotted in Fig. 2 in logarithmic scale. In Fig. 2, a threshold of the problem size at about \(5.8 \times 10^3\) appears for the time cost of ES to exceed the time costs of two matching search algorithms, which shows that the ES algorithm is more efficient below this threshold. The PSO and PSO-CPMS algorithm exhibits an efficiency loss when the problem size is below this threshold because a small search space will make it harder for the PSO algorithm to find an initial solution to (21). At bigger problem sizes, three lines in Fig. 2 tend to share the same slope, and from the biases between each other, it is reflected that the PSO and CPMS algorithm will save time by 64% with respect to ES and will save time by 42%, i.e., about \((1 - 1/\gamma)/2\), with respect to SS-MS.

The rotational and positional parameters \([30]\) of these pulsars are provided by the ATNF pulsar catalog \([31]\).

The position vector is set to \(x = [2.6, -2.6, 2.6]^T \times 10^7\) m, so the true ambiguity of the navigation pulsars is \(m_n = [1, 0, 1]^T\). For the PSO-CPMS algorithm, the number of particles is set to \(S = 40\) and the maximum iteration limit is set to \(N_l = 10\). There are two principles for configuring the \(\gamma\) parameter: 1) It should ensure the PSO algorithm to find an initial solution for (21) within \(N_l\) iterations at a high probability. 2) If the first principle is satisfied, it should be set as large as possible. We choose seven maximum distance \(\rho_m\) values (in \(10^7\) m) 5, 10, 15, 30, 50, 100, and 200 to evaluate the search algorithms at different orders of problem sizes. Via numerical tests, \(\gamma\) is configured to be 3, 3, 4, 6, 8, 8, and 8, respectively, corresponding to the seven \(\rho_m\) values.

The matching search pattern sizes against different problem sizes for the SS-MS algorithm and the PSO-CPMS algorithm are compared in Table I. It is shown in Table I that \(|V_1|\) is about \(2/(1 + 1/\gamma)\) times larger than \(|V_2|\), which indicates that the new algorithm has efficiently decreased the size of the matching search pattern by introducing the parameter \(\gamma\).

For the three search algorithms, the AR process is independently executed for 500 times at each problem size. For every execution, the measurement \(y\) is generated randomly from the variance matrix \(R\). The significance level for every test pulsar is set to \(\alpha = 0.05\). The success probability and related parameters are calculated and listed in Table II, where \(N\) denotes the average number of used test pulsars, \(P_L = (1 - \alpha)^N\) provides a lower boundary of the success probability, and \(P_{\text{ss}}\) is the calculated success probability. Table II shows the following points:

1) The three search algorithms do not differ from each other significantly in success probability for the same significance level.
2) At small problem sizes, AR can be done by one test pulsar and there is \(P_{\text{ss}} \approx P_L\).
3) More test pulsars are needed as the problem size increases and there exists \(P_{\text{ss}} > P_L\) as is expected.
VII. CONCLUSION

In this paper, the IAR problem in PNAV is fully investigated. By means of hypothesis testing, the LAS model is constructed as the fundamental for the search algorithm design. The significance level $\alpha$ is used to control the lower boundary of the success probability for AR. Two existing useful search algorithms are reinterpreted under the LAS model. A new search algorithm based on the matching search technique is also proposed, in which the parameter $\gamma$ is introduced to compress the matching search pattern. Simulation results validate the effectiveness and efficiency of the new algorithm, which shows superiority in performance especially at bigger problem sizes. The main contributions of this paper might be reemphasized as the following points:

1) Theoretical models are built to interpret the IAR problem.
2) With the acceptance space model, the success probability can be exactly controlled.
3) The IAR time cost is saved by $(1 - 1/\gamma)/2$ (nearly a half when $\gamma$ is much greater than one) with respect to the current best algorithm.

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On modeling and pulse phase estimation of X-ray pulsars
Liangwei Huang was born in Yangzhong, China, in 1981. He received the B.S. degree and the M.S. degree both in flight vehicle design and engineering from Northwestern Polytechnical University, Xi’an, China, in 2003 and 2006, respectively, and the Ph.D. degree in automation from Tsinghua University, Beijing, China, in 2013. From 2006 to 2008, he was a Research Assistant with the Shenzhen Graduate School of Harbin Institute of Technology, Shenzhen, China. Since 2013, he has been a Research Associate with the Qian Xuesen Laboratory of Space Technology, Beijing. His current research interests include pulsar-based navigation.

Qingqing Lin was born in 1987. She received the B.S. and Ph.D. degrees in information and communication engineering from Harbin Engineering University, Harbin, China, in 2009 and 2013, respectively. Since 2013, she has been a Research Assistant with the Qian Xuesen Laboratory of Space Technology, Beijing, China. Her research interests include array signal processing, wireless sensor networks, and pulsar-based navigation.
Xinyuan Zhang was born in 1984. He received the B.S. degree in electrical information engineering and the Ph.D. degree in communication and information system from Beihang University, Beijing, China, in 2007 and 2013, respectively. Since 2013, he has been a Postdoctoral Research Assistant with the Qian Xuesen Laboratory of Space Technology, Beijing, China. His research interests include GNSS integrity and signal processing in pulsar-based navigation.

Ping Shuai was born in Liupanshui, China, in 1971. He received the B.S. degree in engineering surveying from the Wuhan Technical University of Surveying and Mapping, Wuhan, China, in 1991, the M.S. degree in geodynamics and tectonic physics from Seismological Institute of Chinese Seismology Bureau, Wuhan, in 1998, and the Ph.D. degree in guidance, navigation, and control from China Aerospace Science & Industry Corporation, Beijing, China, in 2003. From 2003 to 2005, he was a Postdoctoral Research Assistant with the China Academy of Space Technology (CAST), Beijing. From 2005 to 2011, he was a Senior Engineer with the Research and Development Department of CAST. Since 2011, he has been a Senior Researcher with the Qian Xuesen Laboratory of Space Technology, Beijing. His research interests include geodesy, geophysics, integrated navigation system, navigation constellation design, spacecraft autonomous navigation, pulsar-based navigation, and cosmology.