Research Paper

Tensor based geology preserving reservoir parameterization with Higher Order Singular Value Decomposition (HOSVD)

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Article info

Article history:
Received 4 November 2015
Received in revised form
12 May 2016
Accepted 23 May 2016
Available online 14 June 2016

Keywords:
Higher order SVD
Tensor based algorithms
Reservoir parameterization
History matching
Geology preserving parameterization

Abstract

Parameter estimation through robust parameterization techniques has been addressed in many works associated with history matching and inverse problems. Reservoir models are in general complex, nonlinear, and large-scale with respect to the large number of states and unknown parameters. Thus, having a practical approach to replace the original set of highly correlated unknown parameters with non-correlated set of lower dimensionality, that captures the most significant features comparing to the original set, is of high importance. Furthermore, de-correlating system’s parameters while keeping the geological description intact is critical to control the ill-posedness nature of such problems. We introduce the advantages of a new low dimensional parameterization approach for reservoir characterization applications utilizing multilinear algebra based techniques like higher order singular value decomposition (HOSVD). In tensor based approaches like HOSVD, 2D permeability images are treated as they are, i.e., the data structure is kept as it is, whereas in conventional dimensionality reduction algorithms like SVD data has to be vectorized. Hence, compared to classical methods, higher redundancy reduction with less information loss can be achieved through decreasing present redundancies in all dimensions. In other words, HOSVD approximation results in a better compact data representation with respect to least square sense and geological consistency in comparison with classical algorithms. We examined the performance of the proposed parameterization technique against SVD approach on the SPE10 benchmark reservoir model as well as synthetic channelized permeability maps to demonstrate the capability of the proposed method. Moreover, to acquire statistical consistency, we repeat all experiments for a set of 1000 unknown geological samples and provide comparison using RMSE analysis. Results prove that, for a fixed compression ratio, the performance of the proposed approach outperforms that of conventional methods perceptually and in terms of least square measure.

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1. Introduction

In reservoir management, the problems that one faces in controlling petroleum systems are very complex with respect to nonlinearity and uncertainty in their models, which mathematically described by large-scale dynamical systems, at which fine scale discretization and massive computational machinery are very often required to perform simulation and optimization. Furthermore, porous media properties such as conductivity or permeability contain many small scales and uncertainties. For instance, in fractured media, the small scales can be much smaller compared to the field scales. It is computationally expensive to solve forward problems directly, and in particular, it is prohibitively expensive to solve many such forward problems as in the case of uncertainty quantification and controller design. Thus, it is of paramount interest to represent large scale and complex physical models of high dimensionality, e.g., reservoirs, with a reliable one of low dimensionality so that a better forecast of flow and transport within porous media can be obtained through forward reservoir simulation (Oliver and Chen, 2011; Khaninezhad et al., 2014; Aziz and Settari, 1986).

A very challenging problem in reservoir engineering is to estimate petrophysical properties of porous rock such as permeability and porosity, by means of reservoir production history. The dynamic system measurements cannot be forecasted employing static models and thus the process of adjusting geological properties to match the production data, i.e., history matching, became a routine practice (Oliver et al., 2008; Khaninezhad et al., 2014). Having precise predictive reservoir model is of high importance to predict future production profile of reservoir. The main idea is to optimize reservoir production (e.g., hydrocarbons) by delaying
water production and minimizing fluid injection during the reservoir life-cycle. This can be accomplished by setting up a model-based optimization and parameter estimators for proper account of uncertainties. Thus, providing a powerful reservoir characterization method significantly impact the amount of hydrocarbons extracted and, in turn, the volume of global reserves by increasing the recovery factor of conventional reservoirs by only a small fraction. An intensive works have been done on the various aspects of history matching and a solid library of problems and solutions has been created (Oliver and Chen, 2011).

Although, having a reliable and accurate reservoir model is highly tied to providing a fine detailed representation of the reservoir, high resolution model results in a large scale inverse problem both in the parameter space (permeability, porosity) and state-space (pressures, saturations). Therefore, it is of great importance to reparameterize and reduce the dimensionality of the model to acquire efficiency in computational cost. Also, it is beneficial to perform geologically consistent reservoir parameter adjustments as data is being assimilated in the history matching process. This is a daunting task to be performed as one is bounded by the number of data points, i.e., measurements, and very often the inverse problem has more unknowns that can uniquely be estimated from the available data. To this end, having a practical approach to reduce the number of reservoir parameters in order to represent the reservoir model with one of lower dimensionality is of high interest.

A number of studies has been conducted to reduce computational efforts in reservoir simulation and history matching (Elsenbe, 2012; Cardoso and Durlofsky, 2010; van Doren et al., 2006; Scheidt et al., 2009), as well as introducing new applications in multiple point geostatistics (Mariethoz and Caers, 2015; Hu and Chungurova, 2008; Caers, 2001), and reservoir characterization through integrated parameterization (Khaninezhad et al., 2014; Gildin and Afra, 2014). The methodologies basically worked in two fronts: reducing the number of states using model reduction techniques, and reducing the number of unknown parameters in a process known as parameterization (Sarman et al., 2006; Jafarpour, 2013). Among all of these methods, a common approach that is applied to history matching is proper orthogonal decomposition (POD) (Volkwein et al., 2005). In POD, it is necessary to carry out an eigen-decomposition of the random field covariance matrix which is expensive for large models. Furthermore due to the vectorization of the snapshots data in the POD computation, many features may be lost in the reduced space (Belzen and Weiland, 2008; Kolda and Bader, 2009; Lathauwer et al., 2000a).

Here, we introduce a tensor based approach by means of multilinear algebra, namely the higher order singular value decomposition (HOSVD) (Kolda and Bader, 2009; Lathauwer et al., 2000a; Afra and Gildin, 2013; Afra et al., 2014), to reduce high dimensionality of the reservoir property space. HOSVD based techniques have been addressed in literature for many different applications including image processing (Rajawe et al., 2013; Papy et al., 2009), face recognition (Vasilescu and Terzopoulos, 2002), data analysis (Cichocki et al., 2014; Wang and Ahuja, 2005), among others. The application of HOSVD to flow problems has been minimal due to the lack of fast and reliable tools for the decomposition computations. In recent years, however, new tools and algorithms have been developed that show promising results in large-scale multilinear algebra computations (Bader et al., 2012; Sorber et al., 2013). We treat property ensembles as a high-dimensional tensor, and by means of tensor algebra (HOSVD) we show that the reduced models preserve better geometric features keeping them more intact during the reduced basis computations. This plays a pivotal role in reservoir history matching as the properties in consideration have geological meaning. In other words, HOSVD has the ability to reduce the size of the model and reconstruct it accurately while keeping higher order statistical information and geological characteristics of the reservoir model. Conventional parameterization algorithms need to vectorize original replicates to obtain basis in process of dimensionality reduction which may result in losing geological consistency. In HOSVD, we take the snapshots in a 2D or 3D approach and stack them up into a tensor form and by tensor decomposition reduced bases are obtained. One of the advantages to the tensor based approaches is their capability to capture redundancy in all modes, i.e., dimensions, unlike classic SVD which only captures redundancy and noise in one dimension. Eager reader could refer to authors’ other published work on integrating HOSVD based parameterization into history matching process (Gildin and Afra, 2014).

2. Methodology

This section briefly states the parameterization problem and explains essential tools for performing permeability parameterization in the present work.

2.1. Parameterization

Reduction in the number of gridblocks in description of permeability information is of central importance in history matching problems. Indeed, model reduction in this case can be achieved by reparameterizing the permeability field or transforming it to a lower dimensional space by means of permeability parameterization. This can be analogous to treating the permeability field as an image compression problem. A map from space of correlated variables into an uncorrelated space of lower dimension is known as multilinear analysis. For image compression purposes, multilinear techniques are applied to a single known image and the compressed form of the image is obtained by truncating basis in the lower dimension space. A significant task in image processing procedures is to preserve important features of the image along with keeping best basis and achieving an efficient compression ratio.

Parameterization techniques have been studied in subsurface modeling since the 1960s in the form of zonation. In a broad sense, parameterization can be divided into two main groups (Jafarpour, 2013): spatial and transform domain. In the spatial case, one aims at identifying spatial regions (zones) that can be thought as homogeneous pieces of the subsurface, and this can be assigned a single constant property during the inversion process. Although it seems logical to find such zones, problems with geologic discontinuities at the boundary of such regions make the implementation difficult. On the other hand, transform domain methods can overcome these issues by means of introducing geological spatial correlations into the parameterization process. Several techniques have been developed in recent years (Jafarpour, 2013), including the classical principal component analysis, discrete cosine transform (DCT), and the discrete wavelet transform. In all of these cases, the objective is to find an optimal basis selection that can explain the main variability in the parameter fields in a predefined norm. In this paper, we will address some of the shortcomings of the PCA approach.

In the parameterization by PCA case, one aims to reduce dimensionality of the parameter space, which defines the geological description of the reservoir, using the SVD operator. One of the main drawbacks of the SVD framework is the fact that both states and the geological parameters need to be vectorized for the computations of what is known as the snapshot matrix. This, in turn, leads to the loss of geological continuity when the parameters are reconstructed for the reservoir model simulation. We will address this issue by extending the projection framework...
using HOSVD techniques (Afra et al., 2014; Gildin and Afra, 2014),
defined in the next section.

2.1.1. Higher order SVD

One can decompose a \((I_1 \times I_2 \times \cdots \times I_N)\)-order tensor in the si-
milar fashion as in the singular value decomposition for matrices. In fact, every \((I_1 \times I_2 \times \cdots \times I_N)\)-order tensor can be expressed as the
following tensor product:

\[
\mathcal{T} = C U_1^{m_1} U_2^{m_2} \cdots U_N^{m_N}
\]

wherein, \(C\) is an all-orthogonal and ordered tensor referred to as
the core tensor, which represents coefficients matrix in regular SVD,
and \(U_i^{m_i}\)'s are the bases (factor) matrices (which are usually or-
thogonal) and can be thought of as the principal components in
each mode. Moreover, one could derive a matrix representation
form of HOSVD as follows:

\[
\mathcal{T} = U_1^{m_1} C_0 \big(U_1^{m_1+1} \otimes U_2^{m_2+1} \otimes \cdots \otimes U_N^{m_N+1}\big)
\]

in which \(\otimes\) denotes Kronecker product of two matrices. Due to the
orthogonality property, HOSVD is essentially unique. In summary,
computing HOSVD of a tensor is basically given by Eq.(2). A sa-
matter of fact, the \(n\)-mode singular matrix \(U_i\) can simply be found
by calculating the left singular matrix of unfolded form of \(\mathcal{T}\).
Thus, computing HOSVD is equivalent to reapplying SVD, \(N\) times.
Eventually, using the Eq. (2), one could simply compute the core
tensor \(S\).

2.1.2. Truncated HOSVD

Making the analogy of the approximation of a matrix by one of
lower rank using SVD, one can come up with a truncated HOSVD is
similar fashion. Indeed, in order to reduce the order of a given
larger singular values in SVD. Mathematically, given the tensor
\(\mathcal{T} \in R^{I_1 \times I_2 \times \cdots \times I_N}\), we seek to find

\[
\min_{\hat{\mathcal{T}}} \| \mathcal{T} - \hat{\mathcal{T}} \|_F^2
\]

wherein, \(\hat{\mathcal{T}} \in R^{I_1' \times I_2' \times \cdots \times I_N'}\) is the approximated tensor of the order of
\((r_1, r_2, \ldots, r_N)\) which \(r_i < I_i\). Therefore, the problem of finding the
larger mode-\(n\) singular values of a tensor turns into the problem of
finding the best lower rank approximation of a tensor which is an
optimization problem. Unlike the regular SVD, this problem does
not have a closed form solution. A very common approach to ob-
tain a truncated HOSVD is to use the alternating least squares
(ALS) method, proposed by Kroonenberg and Leeuw (1980) and
Bader et al. (2012). Algorithm 1 provides a detailed and step by
step illustration of the procedure that has been developed in the
present work. Fig. 1 depicts HOSVD algorithm pipeline as well.

Algorithm 1. Re-parameterization through HOSVD.

1: procedure 1. TRAINING AND TEST SAMPLE GENERATOR
2: Initialization;
3: Load Static data and geological continuity characteristics
(e.g., variogram etc.)
4: Two-point geostatistical permeability maps \(\rightarrow\) call SGSIM subroutine;
5: Multiple-point geostatistical permeability maps \(\rightarrow\) call SNESIM subroutine;
6: procedure 2. PARAMETERIZATION
7: MATLAB code: \([\text{coreTensor Basis}] = \text{hosvd_als}(\text{permMaps, reducedRank})\)
8: INPUT: \(m\)-order tensor \(\mathcal{T}\)
9: OUTPUT: Core tensor \(C\) and matrices \(U_1, U_2, \ldots, U_M\)
10: Initiate \(U_0, U_1, \ldots, U_M\);
11: while not converged do
12: \(U_i = \mathcal{T} \times U_{i-1}^{m_1} \times \cdots \times U_{m-1}^{m_1} \times U_{m}^{m_1} \times \cdots \times U_M^{m_1}\);
13: \(\max_{i=0}^{M} \| \text{U_i}^{m_1} \times \text{U_i}^{m_2} \times \cdots \times \text{U_i}^{m_N} \|_F\)
14: \(\text{U_i}^{m_1} = f\left(\left(\frac{\text{U_i}^{m_1}}{\max}\right)\right)\)
15: end while
16: procedure 3. MAIN–FORWARD MODEL SIMULATION
17: Compute space basis using training set \(\rightarrow\) call procedure
18: \(\text{HOSVD}(\cdot)\)
19: for Test Set do

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig_1}
\caption{HOSVD parameterization algorithm workflow.}
\end{figure}
3. Experiment description

Having a reliable compression method is paramount in history matching. However, it is not sufficient since, as it was mentioned previously, the permeability image in reservoir history matching problem is unknown. The other important characteristic of a useful compression method for reservoir history matching is the capability of the basis to reconstruct an unknown image with optimum coefficients. In the present experiments, the basis are obtained using prior knowledge of the reservoir permeability. In other words, we use a set of training samples to compute the required set of basis that is necessary for reconstruction and estimation. Using the basis from the reduced dimension space allow us to do both estimation and dimensionality reduction of the unknown permeability maps at the same time. Here, we repeated the experiments for all samples to validate our results and the corresponding root mean square error of the probability density functions, namely RMSE of PDFs are discussed later. Furthermore, an appropriate permeability parameterization and compression method not only has to yield consistent permeability maps but also should result in the similar response of the highly nonlinear and complex reservoir system.

3.1. Statistical analysis

To analyze the reconstruction performance of the SVD and HOSVD we calculate the observation root mean square error (RMSE) of each experiment. RMSE of jth observation (each well’s BHP, oil or water rate) is defined as:

\[ \text{RMSE}_j = \left( \frac{1}{N} \sum_{i=1}^{N} (\hat{d}_j - d_j^i)^2 \right)^{1/2} \]  

(4)

where \( i \) is the time step index. Also, the original and estimated observation are represented by \( d \) and \( \hat{d} \), respectively. In our computations, we do not subtract the mean of the snapshots. It has been shown that this may not make a difference in the case of history matching (Sarma et al., 2006). Also, we work with the logarithm of the permeabilities, and thus no negative values of permeability is possible in our decomposition.

To represent a single observation RMSE for each experiment simplifies comparison of the permeability reconstruction performance of SVD and HOSVD. Since observation RMSEs have different units, we need to make dimensionless RMSEs. To achieve this purpose we utilized the original observation’s \( L^2 \) norm. Finally, we define the total dimensionless observation RMSE as follows:

\[ \text{RMSE}_{\text{total}} = \sum_j \frac{\text{RMSE}_j}{\| d_j \|_2} \]  

(5)

3.2. Note on the comparison between HOSVD and SVD

Given an ensemble of training images \( A \) consists of \( J \) images of size \( I_x \times I_y \), one can reduce the size of original space to a lower rank utilizing the most significant principal components of the original space. Employing the first \( k \) singular values, representation of the data using PCA consists of \( k \) eigenvectors \( \Phi \in \mathbb{R}^{I_x \times I_y \times k} \) and consequently the number of scalars required to represent the data is \( k(I_x^2 + I_y) \).

For HOSVD approximation, a tensor \( T \in \mathbb{R}^{I_x \times I_y \times k} \) is estimated and by a core tensor \( S \in \mathbb{R}^{I_x \times I_y \times k} \) and three basis matrices, \( U^{(1)} \in \mathbb{R}^{I_x \times I_y}, U^{(2)} \in \mathbb{R}^{I_y \times I_y}, U^{(3)} \in \mathbb{R}^{I_y \times I_y} \). Then \( RI_2R_1 + IR_1 + IR_2 + RI_3 \) scalars required to represent the reduced space employing HOSVD.

In order to conduct a fair performance comparison between SVD and HOSVD, we adopted compression ratio criterion defined as follows:

\[ \beta = \alpha \frac{I_x \times I_y \times 3}{\text{dim}^3} \]  

(6)

where \( \alpha \) is the number of scalars required for representing of the ensemble of images. Table 1 summarizes the discussion for different algorithms. For the sake of simplicity and without loss of generality, we assume \( R_1 = R_2 = R \).

After applying projection the best reduced ranked approximations, e.g., \( (R, R, R) \)-rank, of the original tensor is computed. The value of \( \gamma \) is calculated so that the constant compression ratio assumption is fulfilled.

4. Results and discussion

To explore HOSVD-based parameterization properties and evaluate its capability to represent the feature space though exploiting redundancies in all modes, a comprehensive performance analysis is conducted for two different sets of experiments. All results have been compared to those of classic SVD with respect to RMSE measure and for a fixed compression ratio.

4.1. Experiment 1: Synthetic model

We apply tensor decomposition approach with ensemble of 2D permeability fields to illustrate how this method can provide low-dimensional geologically realistic permeability-field parameterizations suitable for history matching. A synthetic model is generated utilizing multiple-point geostatistical simulation algorithm, namely the single normal equation simulation (SNESIM) (Strebelle et al., 2001). A training image is selected to draw fluvial channel replicates. The training image 2(a) along with two generated samples are illustrated in Fig. 2. This training image has \( 470 \times 470 \times 1 \) pixels and contains two facies types, the first facies is low permeability as a background and the second one is the high permeability channels. The low permeability value is 500 md while the high permeability value is 10,000 md. A training set, consists of permeability realizations, of size 1000 is generated to represent the geological structure. Each realization contains \( 45 \times 45 \) grid blocks. Different shapes and geometry of the channel in

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**Table 1**

Comparison between PCA and HOSVD in data representation factors.

<table>
<thead>
<tr>
<th>Method</th>
<th>Decomposition</th>
<th># of scalars</th>
</tr>
</thead>
<tbody>
<tr>
<td>HOSVD</td>
<td>( \Phi = \hat{S} x_1 (U^{(1)} x_1 U^{(1)}) + \hat{S} x_2 (U^{(2)} x_2 U^{(2)}) + \hat{S} x_3 (U^{(3)} x_3 U^{(3)}) )</td>
<td>PCA</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( k(I_x^2 + I_y) )</td>
</tr>
</tbody>
</table>

---

**Note:**

\( \Phi \) is the optimum coefficient of the basis to reconstruct an unknown image with optimum coefficients. In the present experiments, the basis are obtained using prior knowledge of the reservoir permeability.
these realizations are the major uncertainty in the unknown permeability field.

Each permeability realization is a 45 × 45 map of 30 × 30 × 30 feet grid block size and the reservoir assumed to act as a 2D two-phase fluid flow problem. A water flooding scenario similar to Sarma et al. (2006) and Brouwer et al. (2004) is utilized here to simulate the fluid flow problem. Forward simulations are performed using ECLIPSE (S.T. Corporation, 2010). Overall, we conducted three different sets of forward simulations. In the first setup the ensemble of original permeability samples are used as an input to the simulator. The other two setups are designed and implemented to feed simulator with ensembles of reduced order permeability samples, using SVD and HOSVD respectively, as input. As discussed before, since we utilized 2-fold CV with no repetition, thus, we performed three sets of 4 × 1000 forward reservoir simulations to complete error estimation analysis. A randomly drawn permeability realizations along with corresponding reduced ordered representatives using classic SVD and HOSVD are shown in Fig. 2(b) and (c). All 45 producers are operating with a constant bottom-hole pressure (BHP) of 2900 psi and horizontal wells with 45 ports is considered injecting water with a constant injection rate to secure a total of 1 pore volume of water is injected through the simulation time of 1 year. As stated previously, porosity is not parameterized in this work and therefore is fixed at 0.22. Table 2 summarizes all information regarding simulation description and reservoir specifications.

We applied HOSVD-ALS algorithm to permeability replicates as a training set to find the basis function employed to reduce the dimension of the original images’ space for compression purposes. First, HOSVD-ALS is utilized to provide truncated basis matrices and a coefficient core tensor. Based on the HOSVD-ALS approach, the reduction procedure requires the selection of basis which express the map with higher accuracy, e.g., those which capture more significant features. Here, the following scenario is performed to obtain

![Image](https://example.com/image.jpg)
re-parameterized permeability representations using both SVD and HOSVD. In parameterization with the classic SVD, all samples are vectorized and stacked into a matrix namely the training set, and SVD is performed to find the space basis required for the reconstruction step. Then, the singular vectors associated with the largest 55 singular values of the covariance matrix, which conserve the most of the energy, are selected as reduced order space’s basis. In the SVD case we used those selected basis to compute reduced order representation of the original permeability maps to reduce the dimensionality of original space. This is not the case in the HOSVD, as we keep the snapshots as it is. For the HOSVD parameterization, the best $(15, 15, \gamma) - rank$ representation of the original tensor is approximated and then Kronecker product is performed in all simulations to compute required coefficients for the reconstruction step. Utilizing truncated basis results in a compressed reconstructed version of known permeability maps. As mentioned before, all experiments are run for a fixed compression ratio. To conduct a fair comparison between classic SVD and HOSVD, $\gamma$ is calculated as 135 to satisfy fixed compression ratio of 0.0822.

Fig. 2 (b) and (c) also show that HOSVD not only outperforms classic SVD in the measure of RMSE, but also overcomes its representation perceptually, i.e., if we consider permeability realizations as images. One can see that HOSVD-based parameterization is able to better capture the high and low permeability edges compared to the SVD-based method. It is evident from the results that HOSVD-ALS has a promising power to compress an image while preserving geological features along with spacial characteristics of permeability maps. In addition to preserving the most important features, the low computational cost is the other significant property of HOSVD. A number of intensive studies in machine learning applications have addressed the computational cost of tensor based algorithms and many efficient techniques, i.e., in computational cost and memory sense, to compute Rank-$R$ approximation of a tensor (Lathauwer et al., 2000a, 2000b; Wang and Ahuja, 2005; Rajwade et al., 2013). In fact, in practical reservoir parameterization problems with system of order of millions, low computational cost and the ability to reduce to a lower rank become important. Eager reader can refer to Afra and Gildin (2013) and Afra et al. (2014) for more detailed analysis and discussion about the capabilities of HOSVD parameterization.

Comparing oil production rate curves in Fig. 3(a) illustrates the advantage of using HOSVD over the classic SVD even more particularly once the SVD response fails to follow the trend of original response. Fig. 3(b) presents water production rates. Interestingly, the SVD based parameterization causes an earlier breakthrough comparing to the original response and that of HOSVD. An important characteristic of a powerful compression method for reservoir history matching is the capability of the basis to reconstruct an unknown image with optimum coefficients. In the present experiments, basis are obtained using prior knowledge of

**Fig. 3.** Production curves for synthetic $45 \times 45$ Example. System responses are plotted for original map as well as classic SVD and HOSVD based parameterized maps: (a) Oil production rate curves of all producing wells. (b) Water production rate curves of all producing wells for the second sample.

**Fig. 4.** Probability density functions (PDF) of expected error, i.e., a training-set based error estimation like cross validation used for synthetic $45 \times 45$ Example: (a) PDFs of permeability reconstruction RMSE for SVD and HOSVD; (b) PDFs of total dimensionless observation RMSE for SVD and HOSVD.
reservoir permeability. In other words, we use a set of training samples to compute required set of basis that is necessary for reconstruction and estimation part. As mentioned before, we repeated this procedure for all samples utilizing 2-fold CV with no repetition, also known as holdout method, to validate our results and corresponding RMSE PDFs are discussed in the next section. Here, we presented a randomly drawn samples to compare the results. The reader can refer to (Webb, 2003) for more details on cross validation techniques.

In order to statistically demonstrate the advantage of HOSVD over SVD we applied both methods to a set of size 2000 including permeability realizations. In each step of cross validation error estimation, a set of size 1000 samples is left out to be used as the test set. In order to reduce the variability associated with the cross validation method, we repeated this folding process 10 times with different folds. We also repeated the forward reservoir simulation for all permeability samples within the training and test set to statistically compare the performance of HOSVD-based and classic SVD-based parameterization in reproducing the original response of the system. Fig. 4(a) which includes permeability RMSE PDFs clearly shows that HOSVD results in considerably less reconstruction error than SVD in the parameterization process. This convey the fact that HOSVD-ALS has a promising power to compress a permeability image while preserving geological features along with spatial characteristics of the permeability map. Fig. 4(b) express the probability density function of total dimensionless observation RMSE using the results of all forward simulations. Corresponding statistics are concluded in Table 3.

4.2. Experiment 2: SPE10 model

To prove the advantages of the proposed parameterization method, we employ the top 5 layers of SPE10 benchmark (Christie and Blunt, 2001). A two-point geostatistical simulation algorithm is utilized to generate 1000 permeability realizations. Here, to represent the geological structure, we incorporated patterns from top 5 layers of SPE10 to an exponential variogram model with correlation length properties equal to 100 and 80 in the x and y directions, respectively. Moreover, a proper permeability covariance matrix is constructed in order to assure each map includes the desired permeability range in the top 5 layers of SPE10 model. Each permeability realization is a $60 \times 220 \times 5$ map of $33 \times 33 \times 33$ feet grid block size. We run ECLIPSE reservoir simulator for a 3D two-phase (oil and water) black oil during 360 days or 6 time intervals of 60 days. All the simulation setup and parameterization procedure are done in a similar fashion to the synthetic example. Again, we conducted three distinct sets of forward simulation experiments for original, classic SVD, and HOSVD parameterized maps. A randomly drawn true permeability realization along with corresponding reduced ordered representative using classic SVD and HOSVD is shown in Fig. 5. Eight producers are operating along with 15 injectors constructing a network of eight inverted 5 spot configuration all in a waterflooding scenario. As stated previously, porosity is not reparameterized in this work and therefore assumed to be distributed uniformly and fixed at

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>Median</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>45x45</td>
<td>HOSVD</td>
<td>10.53</td>
<td>0.1245</td>
</tr>
<tr>
<td></td>
<td>Classic SVD</td>
<td>57.1368</td>
<td>0.6482</td>
</tr>
<tr>
<td>SPE10</td>
<td>HOSVD</td>
<td>0.3437</td>
<td>0.3274</td>
</tr>
<tr>
<td></td>
<td>Classic SVD</td>
<td>0.7382</td>
<td>0.7204</td>
</tr>
</tbody>
</table>

Fig. 5. First permeability sample along with its reconstructed versions by SVD and HOSVD. SPE10 example.
0.22. Furthermore, 1 pore volume of water is injected through the simulation time of 1 year.

Figs. 5–8 summarize all results for a randomly picked replicate out of SPE10 ensembles. We plotted bottom hole pressure curves for 9 selected injectors as well as oil and water production rates for 6 producers for the sake of simplicity to compare results and values for the selected samples. Similar to the synthetic example, Fig. 5 illustrates that HOSVD not only outperforms SVD in terms of permeability RMSE, but also overcomes it in perceptual sense. Moreover, the capability of reducing rank in all modes is assisting HOSVD to become more efficient in the manner of memory and computational costs as well even more than the previous example due to the higher dimension nature of the second example. Fig. 6 shows the bottom hole pressure curves for the original samples and corresponding HOSVD-based and SVD-based reparameterize versions. In all injectors the corresponding results of HOSVD-based parameterization tends to follow the trend of original response both in shape and values. One may see that in injector number 3, 12 and 13 the HOSVD response perfectly fits to the true response. In fact, it is obvious from the results that utilizing HOSVD-based parameterization not only result in better reduced order versions perceptually and in terms of RMSE, but also produces similar responses to that of original maps which is more important in our purpose.

Oil production rates are depicted in Fig. 7(a). One can simply compare the HOSVD method results with that of SVD method. Again, it is apparent form the results that even for the simpler permeability map with a non-complex trend, HOSVD-based parameterization curves outperforms those of classic SVD parameterization. For instance, in the producers 1, 2 and 7, the SVD-based results are way off in values and trend comparing to those of original and HOSVD-based. Fig. 7(b) displays water production rate curves as well. In contrast to the first example, we do not observe breakthrough in any of the producers during the simulation period. Although, no breakthrough happened in the experiments, the difference between HOSVD and classic SVD is noticeable specifically for more complex trends in permeability maps.

In order to statistically demonstrate the advantage of HOSVD over SVD we applied both methods to 2000 permeability samples employing 2-fold CV with no repetition. All experiments are run for a fixed compression ratio. To conduct a fair comparison between classic SVD and HOSVD, $\gamma$ is calculated as 634 to satisfy fixed compression ratio of 0.0592. In fact, the best $(15, 15, 634)$-rank approximation of the tensor is computed through HOSVD-ALS and the 55 most significant singular values are picked for regular SVD. Fig. 8(a) which includes permeability RMSE PDFs, clearly shows that HOSVD results in considerably less reconstruction error than SVD in parameterization process. This conveys the fact that HOSVD-ALS has a promising power to compress an image while preserving geological features along with spacial characteristics of permeability map. Fig. 8(b) express the probability density function of total dimensionless observation RMSE using the results of all forward simulations. It should be noted that 90% of total dimensionless observation RMSE values lay within the shown interval in Figs. 8 and 4. The statistics of the probability density functions corresponding to the SPE10 example summarized in Table 3.

4.3. Running time comparison

Comparison between classic SVD and HOSVD with respect to running time is of high importance in order to evaluate their performance as well as their efficiency. As mentioned previously, we generate two ensemble of 1000 permeability realizations employing two-point geostatistical simulation algorithm (SGSIM) and an exponential variogram with specific assumed covariance and correlation parameters. It should be noted that variogram properties are set independently for $45 \times 45$ and SPE10 cases. The below results are generated as follows. All 1000 samples in each ensembles
are considered as training sets of known permeability maps. Then, we apply classic SVD and HOSVD based parameterization method to the ensembles and observe running time of each scheme first for computing basis and coefficients of the parameter space and second for compression process (reducing the order of each map). The results are summarized in the Table 4. One must note that for SPE10 example, we repeated a similar fashion for each layer of every realizations separately and independently. Then the mean of all running times is reported to acquire statistical consistency. Results show that HOSVD method outperforms classic SVD with respect to running time when computing the basis and coefficients as well as reconstructing step.

5. Conclusion

Reducing the dimensionality of parameter space, e.g., the space of permeability or porosity realizations, is of central interest in
reservoir history matching and simulation specifically when the problem is of very high order (millions of grid blocks). A high reduction in size of parameter space, decreases the number of unknown geological properties that have to be estimated through the reservoir history matching procedure. Furthermore, a daunting task in reservoir characterization is to choose the principal features of the parameter space, i.e., those that conserve most of the energy, in order to improve the estimation. One must note that a powerful parameterization method not only provides better basis, but also does so in a time efficient manner. To this end, having a powerful parameterization method is of high importance in reservoir simulation and inverse problems.

The results of the present work illuminate the promising power of HOSVD to reparameterize and capture the important geological features of the permeability realizations. Moreover, as a powerful parameterization and compression method, HOSVD regenerates similar petrophysical property representations, e.g., permeability. It also reproduces same response of the highly nonlinear and complex reservoir system. In other words, the proposed approach employed in this work can strongly capture all important spatial features and all spatial geological characteristics in an efficient manner with respect to the size of low dimensionality representations space, running time and performance perspectives, comparing to regular SVD as failed to preserve spatial features due to vectorization. Indeed, HOSVD re-parameterization offers a more consistent, e.g., geologically, reduced order representation of the parameter space as one can observe from the RMSE plots. Also, to acquire statistical consistency, all experiments were repeated utilizing 2-fold CV with no repetition. Comparing running time, proves that HOSVD overcomes classic SVD in running time as well. One must note that the provided results can easily be reproduced for the case of porosity re-parameterization. To conclude, the present work indicates that HOSVD provides a promising tool for permeability parameterization in reservoir characterization required for history matching processes.

![Image](87x557 to 519x735)

**Fig. 8.** Probability density functions (PDF) of expected error, i.e., a training-set based error estimation like cross validation used for SPE10 Example: (a) PDFs of permeability reconstruction RMSE for SVD and HOSVD; (b) PDFs of total dimensionless observation RMSE for SVD and HOSVD.

<table>
<thead>
<tr>
<th>Method</th>
<th>Computing time</th>
<th>Reconstructing time</th>
</tr>
</thead>
<tbody>
<tr>
<td>HOSVD</td>
<td>2.0639</td>
<td>37.1983</td>
</tr>
<tr>
<td>Classic SVD</td>
<td>39.5381</td>
<td>456.4611</td>
</tr>
</tbody>
</table>

**Table 4** Running time comparison between HOSVD and classic SVD (in seconds).

**Acknowledgment**

The authors would like to thank the support from the US DoD Army ARO under grant #W911NF-12-1-0206.

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